上海科技大学 ShanghaiTech University
KAVLI ITS Kavli Institute for Theoretical Sciences
中国科学技术大学 University of Science and Technology of China

Quantum Gravity and Cosmology 2024

ShanghaiTech University, Shanghai, China, July 1-5

Alexander Vikman

05.07.2024

Co-funded by the European Union

MŠMT MINISTRY OF EDUCATION, YOUTH AND SPORTS

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PHYSICAL REVIEW LETTERS **128,** 041301 (2022)

Ghosts without Runaway Instabilities

Cédric Deffayet,[1,2,*] Shinji Mukohyama,[3,4,†] and Alexander Vikman[5,‡]

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[2]IHES, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France
[3]Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502 Kyoto, Japan
[4]Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
[5]CEICO–Central European Institute for Cosmology and Fundamental Physics, FZU–Institute of Physics of the Czech Academy of Sciences, Na Slovance 1999/2, 18221 Prague 8, Czech Republic

(Received 26 August 2021; accepted 24 December 2021; published 24 January 2022)

We present a simple class of mechanical models where a canonical degree of freedom interacts with another one with a negative kinetic term, i.e., with a ghost. We prove analytically that the classical motion of the system is completely stable for all initial conditions, notwithstanding that the conserved Hamiltonian is unbounded from below and above. This is fully supported by numerical computations. Systems with negative kinetic terms often appear in modern cosmology, quantum gravity, and high energy physics and are usually deemed as unstable. Our result demonstrates that for mechanical systems this common lore can be too naive and that living with ghosts can be stable.

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JCAP Anniversary Special Issue

Global and local stability for ghosts coupled to positive energy degrees of freedom

Cédric Deffayet,[a] Aaron Held,[b,c] Shinji Mukohyama[d,e] and Alexander Vikman[f]

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[b]*Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany*
[c]*The Princeton Gravity Initiative, Jadwin Hall, Princeton University, Princeton, New Jersey 08544, U.S.A.*
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Talks by:

Damiano Anselmi
Manuel Asorey
Mariam Bouhmadi-Lopez
Luca Buoninfante
Bob Holdom
Roberto Percacci,
Taotao Qiu
Alberto Salvio
Yuhang Yang….

PHYSICAL REVIEW D, VOLUME 65, 103515

Living with ghosts

S. W. Hawking* and Thomas Hertog†
DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom
(Received 27 July 2001; published 9 May 2002)

Perturbation theory for gravity in dimensions greater than two requires higher derivatives in the free action. Higher derivatives seem to lead to ghosts, states with negative norm. We consider a fourth order scalar field theory and show that the problem with ghosts arises because, in the canonical treatment, ϕ and $\Box\phi$ are regarded as two independent variables. Instead, we base quantum theory on a path integral, evaluated in Euclidean space and then Wick rotated to Lorentzian space. The path integral requires that quantum states be specified by the values of ϕ and $\phi_{,\tau}$. To calculate probabilities for observations, one has to trace out over $\phi_{,\tau}$ on the final surface. Hence one loses unitarity, but one can never produce a negative norm state or get a negative probability. It is shown that transition probabilities tend toward those of the second order theory, as the coefficient of the fourth order term in the action tends to zero. Hence unitarity is restored at the low energies that now occur in the universe.

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Not that kind of **Living with ghosts**

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No way to live with ghosts and no way to live without ghosts.

Ilya's Shapiro

Why are we interested in ghosts?

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$$S = \int M_{\rm Pl}^2 R + \alpha R^2 + \beta W_{\mu\nu\sigma\lambda} W^{\mu\nu\sigma\lambda}\,, \quad \text{Weyl tensor} \quad W = \partial\partial g$$

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- Questions related to entropy and thermodynamics
- Is it possible to screen gravity?
- Is it possible to screen the Cosmological Constant or the energy of quantum vacuum?
- Interesting exotic compact objects like wormholes
- Can gravitons be massive? (Boulware–Deser ghost, 1972, dRGT etc.)

Giuseppe Ludovico De la Grange Tournier

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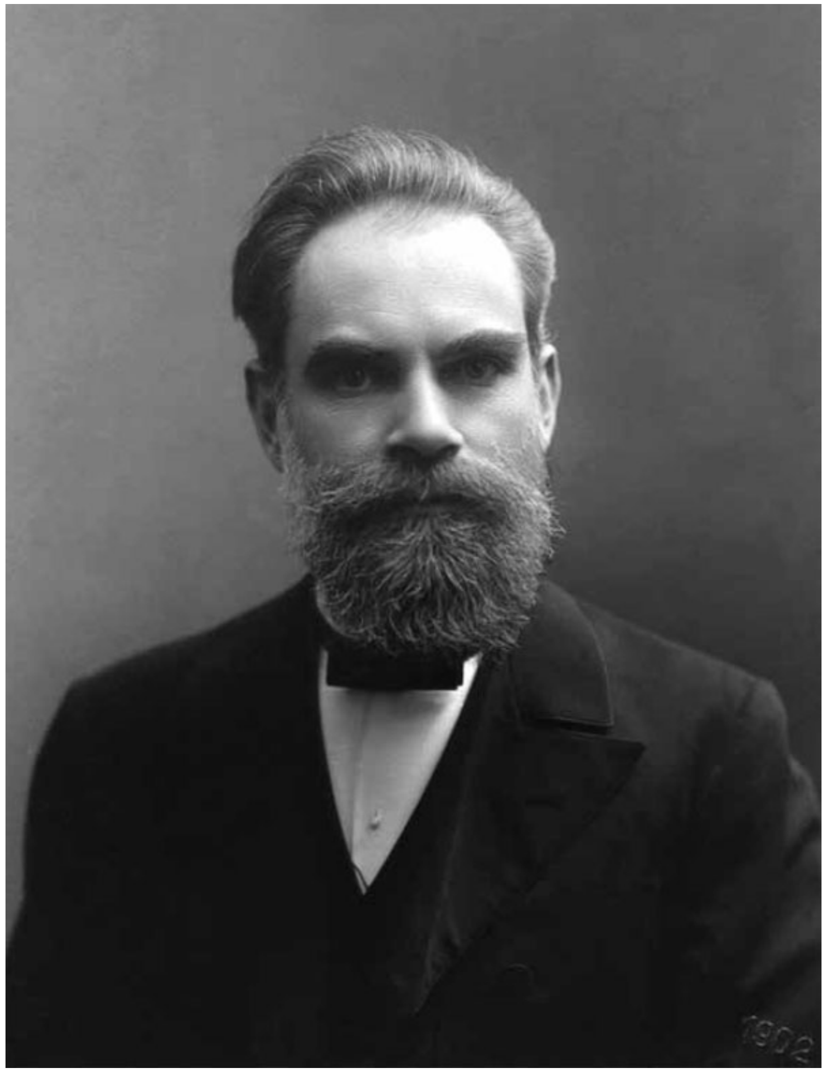
Lagrange Stability

the motion is finite -
is bounded in phase space -
“Global Stability”

Giuseppe Ludovico De la Grange Tournier



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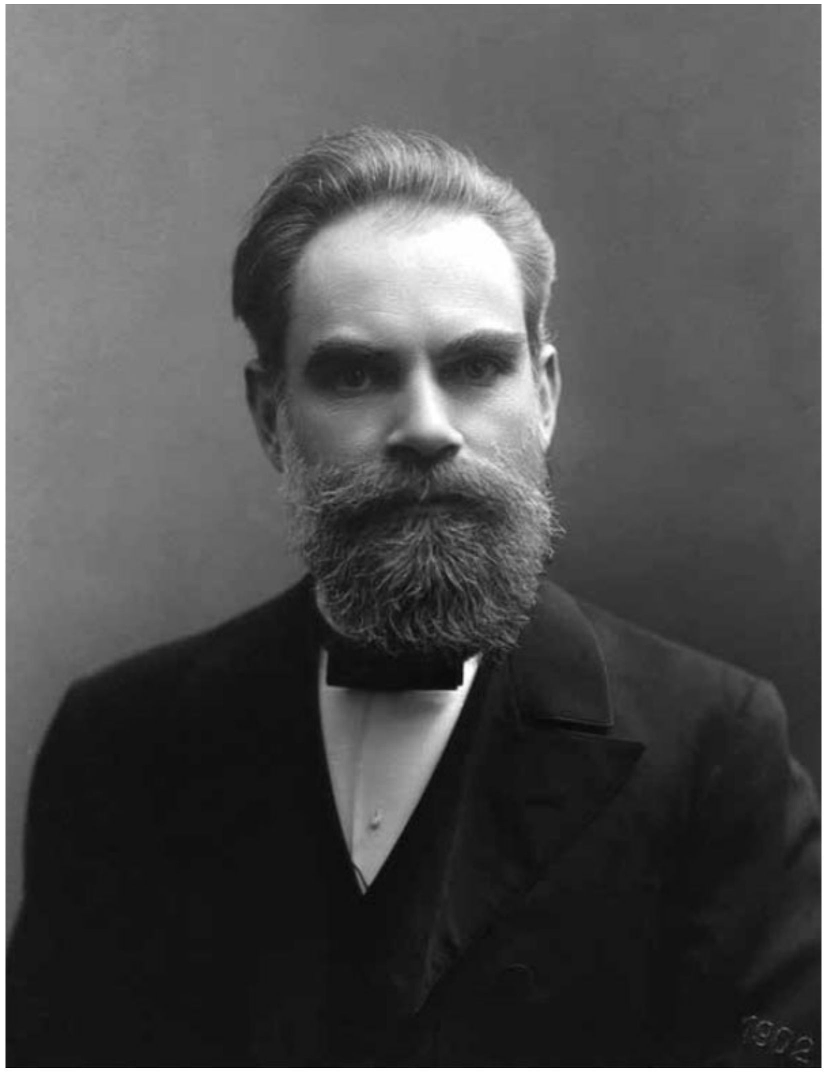

Aleksandr Mikhailovich Lyapunov

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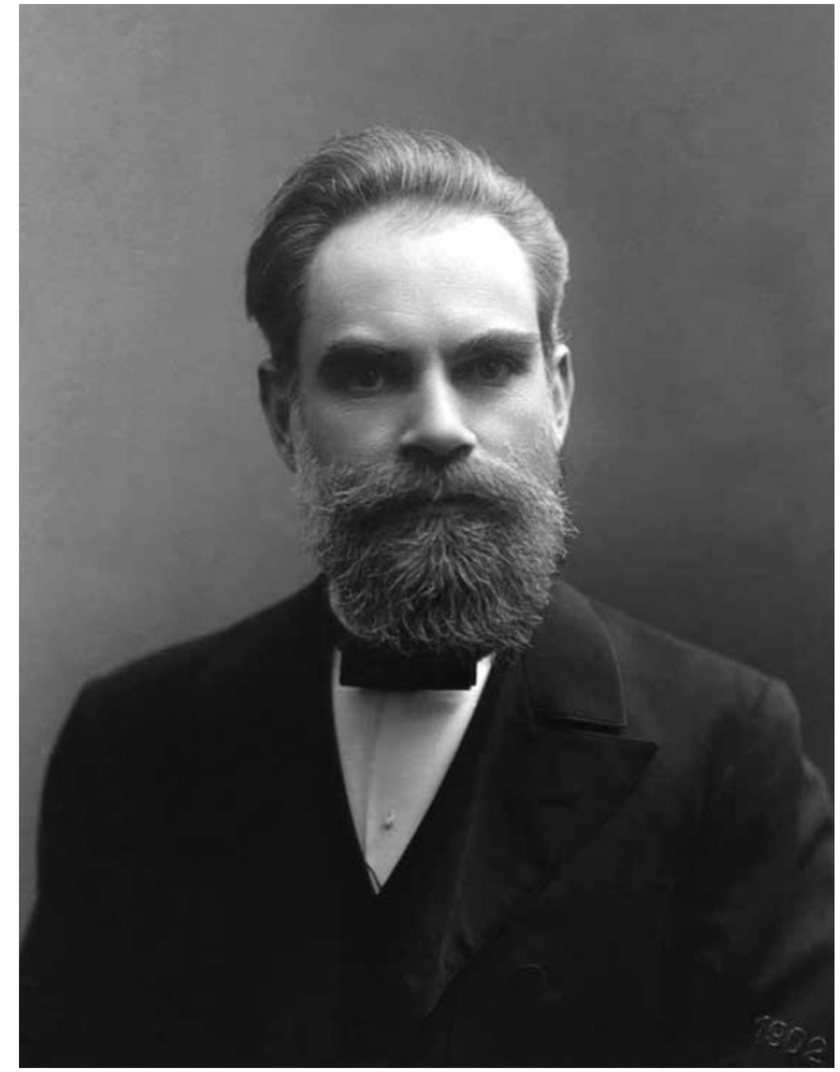

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Lyapunov Stability

solutions starting
"close enough"
(within a distance δ from each other)
remain "close enough" forever
(within a distance ϵ from it).

Ostrogradsky Theorem

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modern version for poor people

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$$\frac{1}{M^2p^2 - p^4}$$

propagator

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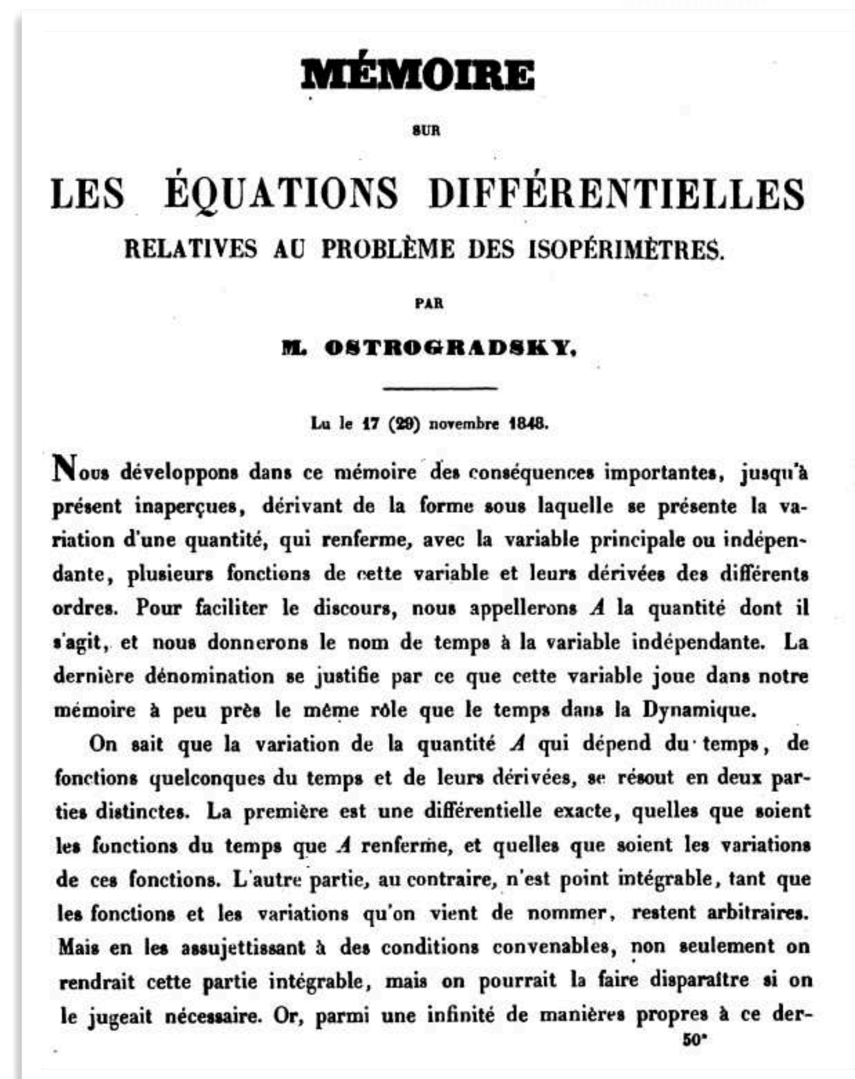

MÉMOIRE

SUR

LES ÉQUATIONS DIFFÉRENTIELLES

RELATIVES AU PROBLÈME DES ISOPÉRIMÈTRES.

PAR

M. OSTROGRADSKY.

Lu le 17 (29) novembre 1848.

Nous développons dans ce mémoire des conséquences importantes, jusqu'à présent inaperçues, dérivant de la forme sous laquelle se présente la variation d'une quantité, qui renferme, avec la variable principale ou indépendante, plusieurs fonctions de cette variable et leurs dérivées des différents ordres. Pour faciliter le discours, nous appellerons A la quantité dont il s'agit, et nous donnerons le nom de temps à la variable indépendante. La dernière dénomination se justifie par ce que cette variable joue dans notre mémoire à peu près le même rôle que le temps dans la Dynamique.

On sait que la variation de la quantité A qui dépend du temps, de fonctions quelconques du temps et de leurs dérivées, se résout en deux parties distinctes. La première est une différentielle exacte, quelles que soient les fonctions du temps que A renferme, et quelles que soient les variations de ces fonctions. L'autre partie, au contraire, n'est point intégrable, tant que les fonctions et les variations qu'on vient de nommer, restent arbitraires. Mais en les assujettissant à des conditions convenables, non seulement on rendrait cette partie intégrable, mais on pourrait la faire disparaître si on le jugeait nécessaire. Or, parmi une infinité de manières propres à ce der-

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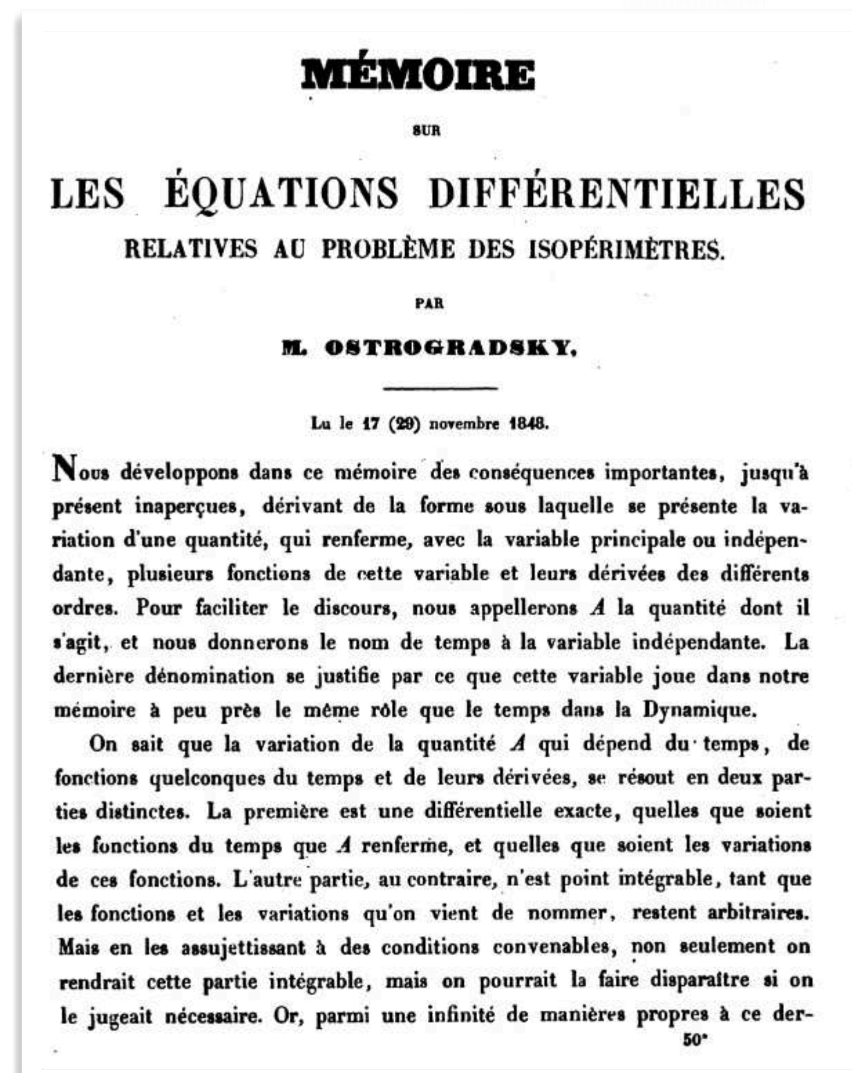

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For Lagrangian $L\left(q, \dot{q}, \ddot{q}\right)$ depending on acceleration $a = \ddot{q}$

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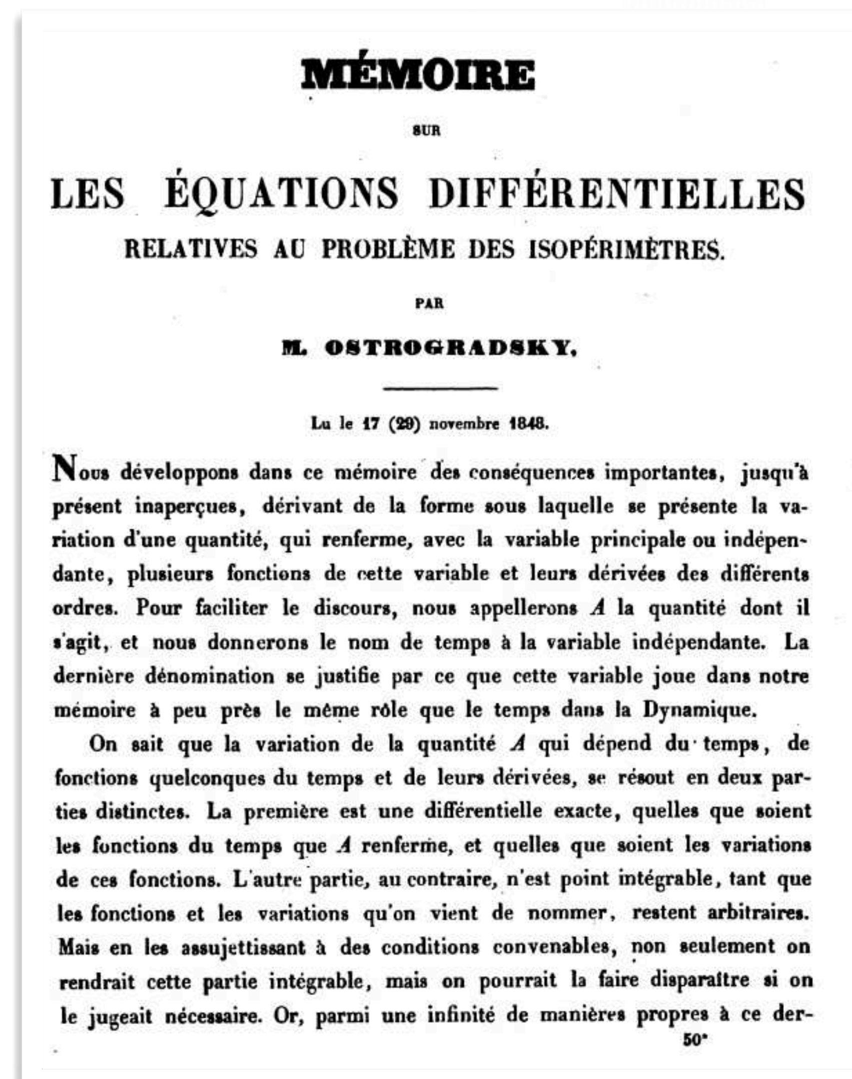

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canonical momentum for $Q_1 = q$

$$P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}$$

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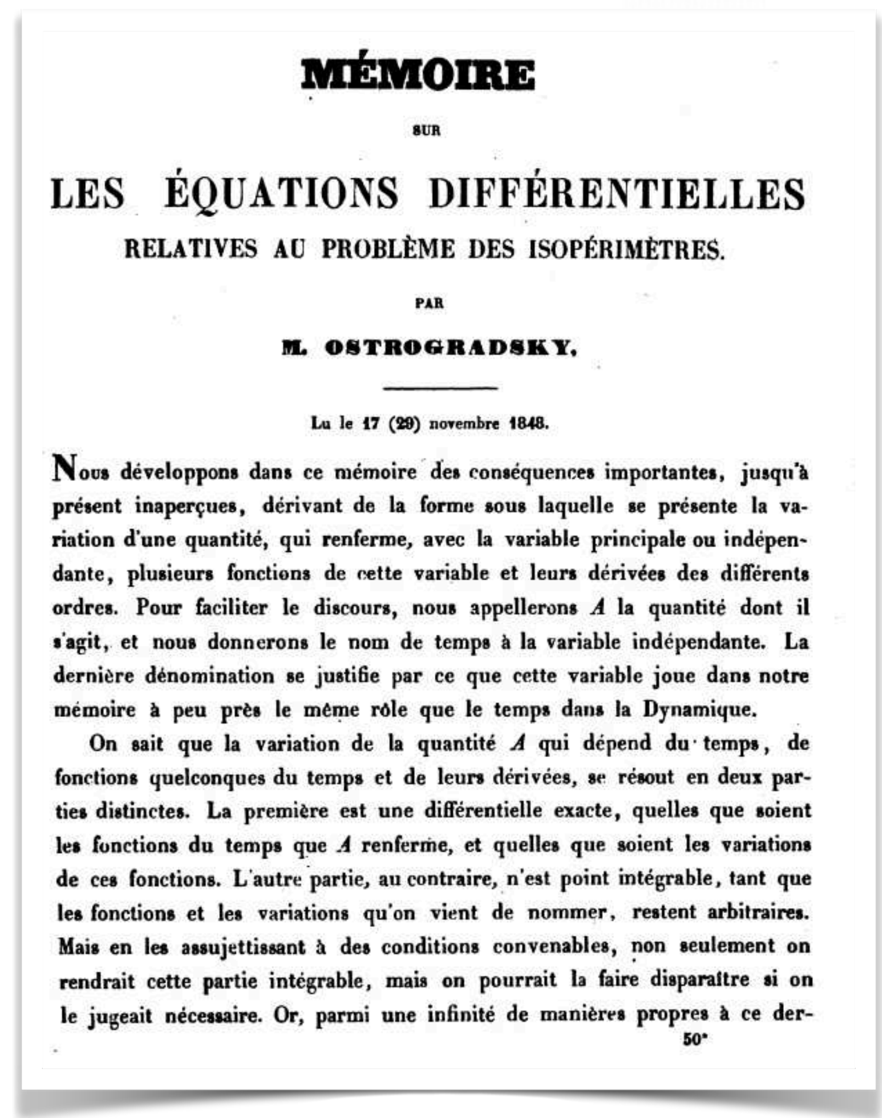

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canonical momentum for $Q_1 = q$

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Ostrogradsky Theorem

modern version for poor people

$$\frac{1}{M^2p^2 - p^4} = \frac{1}{M^2}\left[\frac{1}{p^2} - \frac{1}{p^2 - M^2}\right]$$

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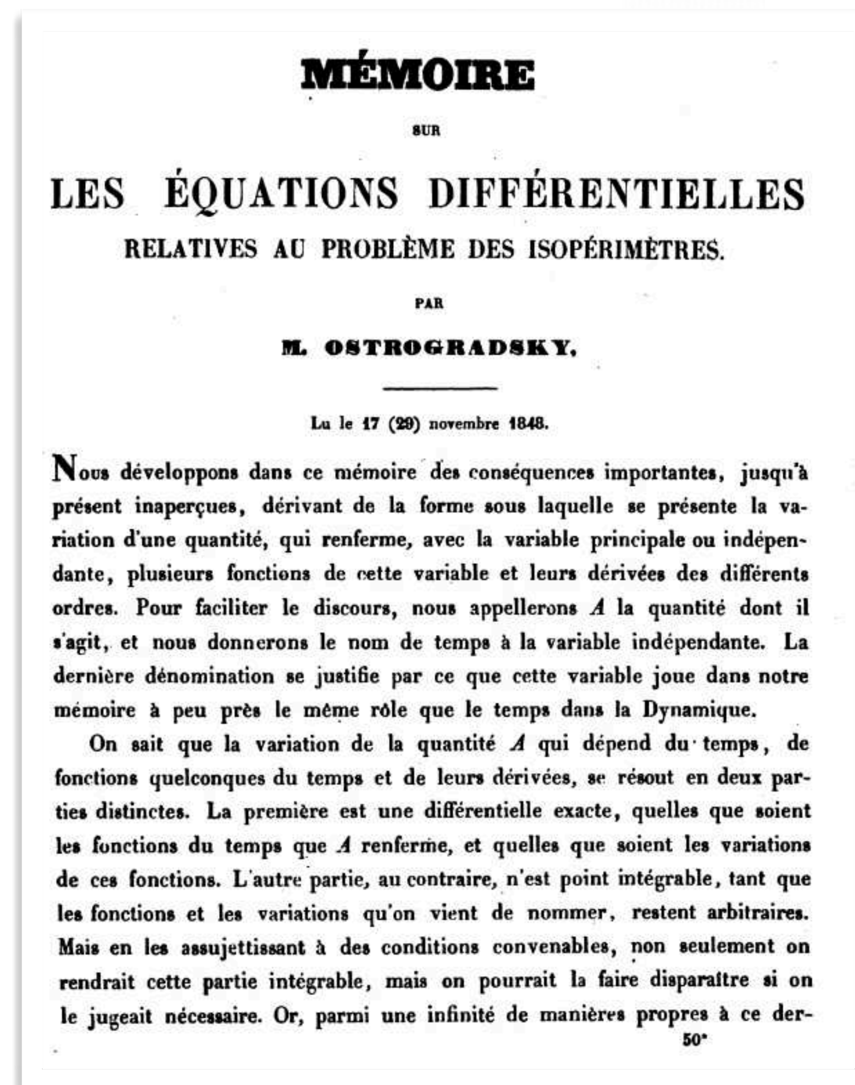
Mikhail Vasilyevich Ostrogradsky



MÉMOIRE

SUR

LES ÉQUATIONS DIFFÉRENTIELLES

RELATIVES AU PROBLÈME DES ISOPÉRIMÈTRES.

PAR

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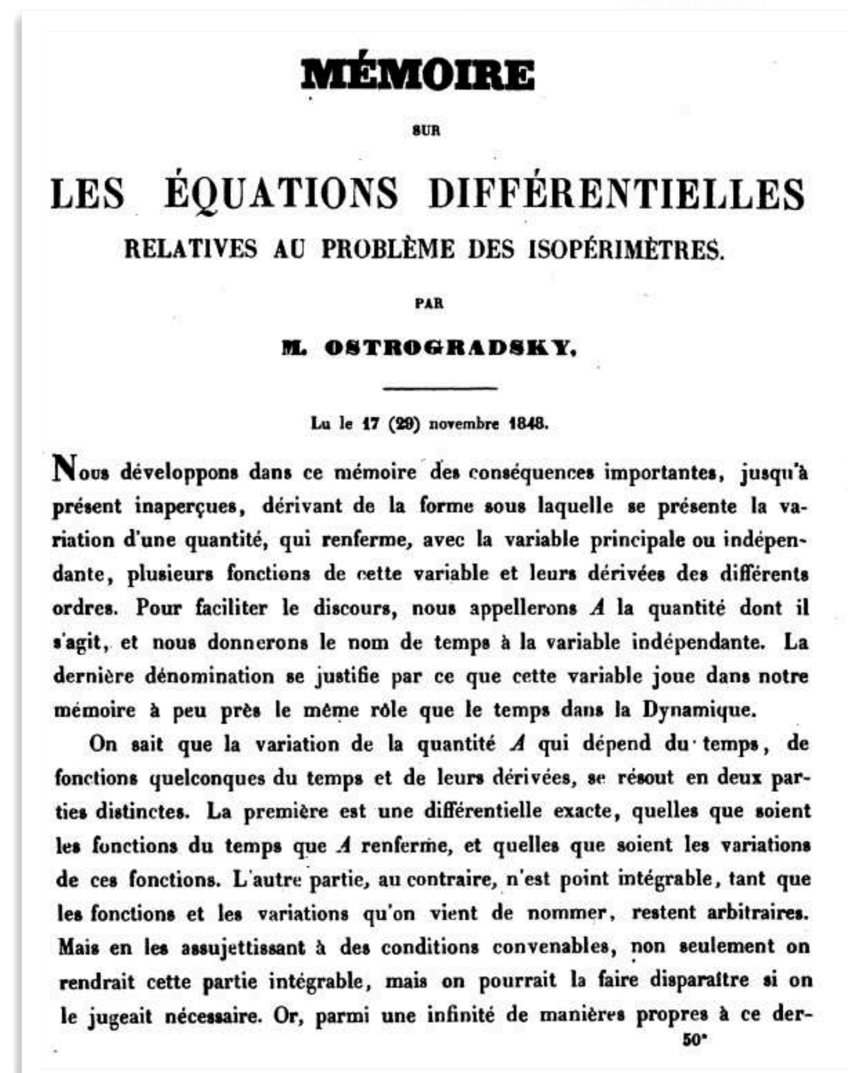
Hamiltonian linear in P_1 - unbounded from above and from below!



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$$L = \frac{1}{2}\left(\dot{q}^2 - q^2 + \alpha \ddot{q}^2\right)$$

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e.g. action for cosmological perturbations

$$S\left[\mathcal{R}\right] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, Z\left(\left(\mathcal{R}'\right)^2 - c_{\mathrm{S}}^2\left(\partial_i\mathcal{R}\right)^2\right)$$

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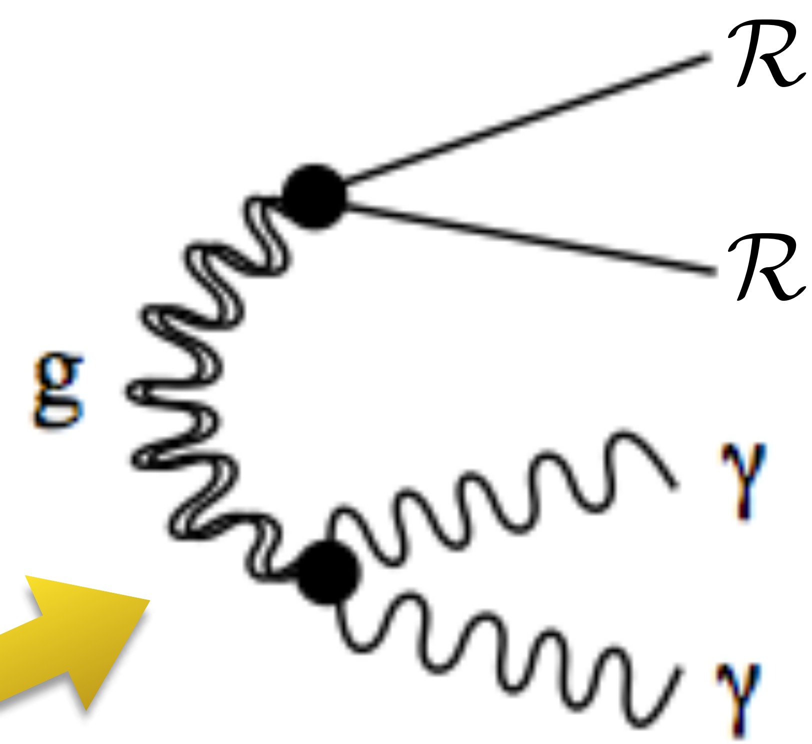

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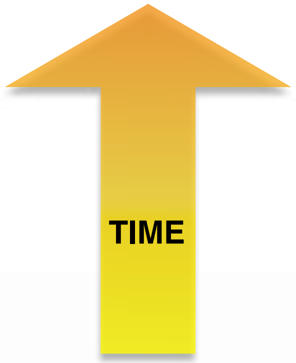
$$\Gamma_{0\to2\gamma2\phi}\sim\frac{\Lambda^{8}}{M_{\mathrm{Pl}}^{4}}$$

Cline, Jeon, Moore, (2003)

Instability

$E_{tot} = E_N + E_G$

Instability



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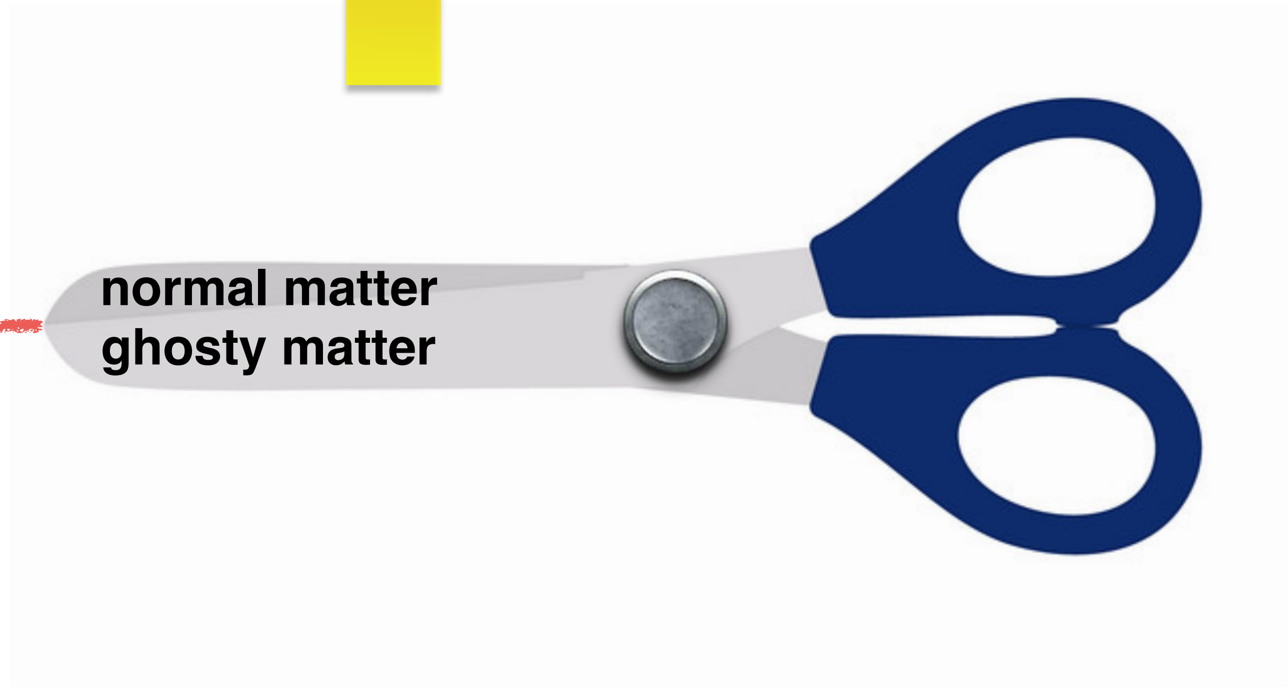

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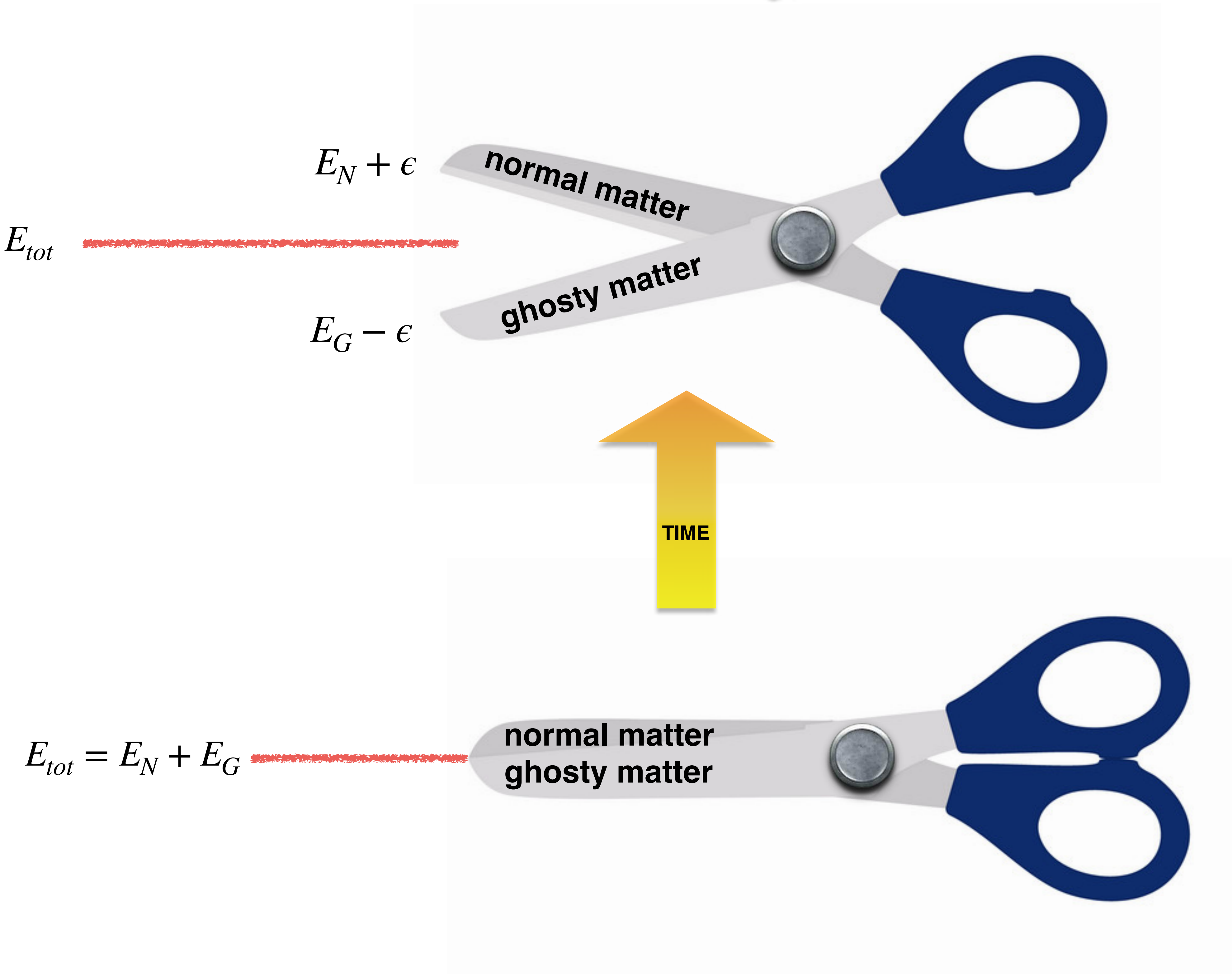

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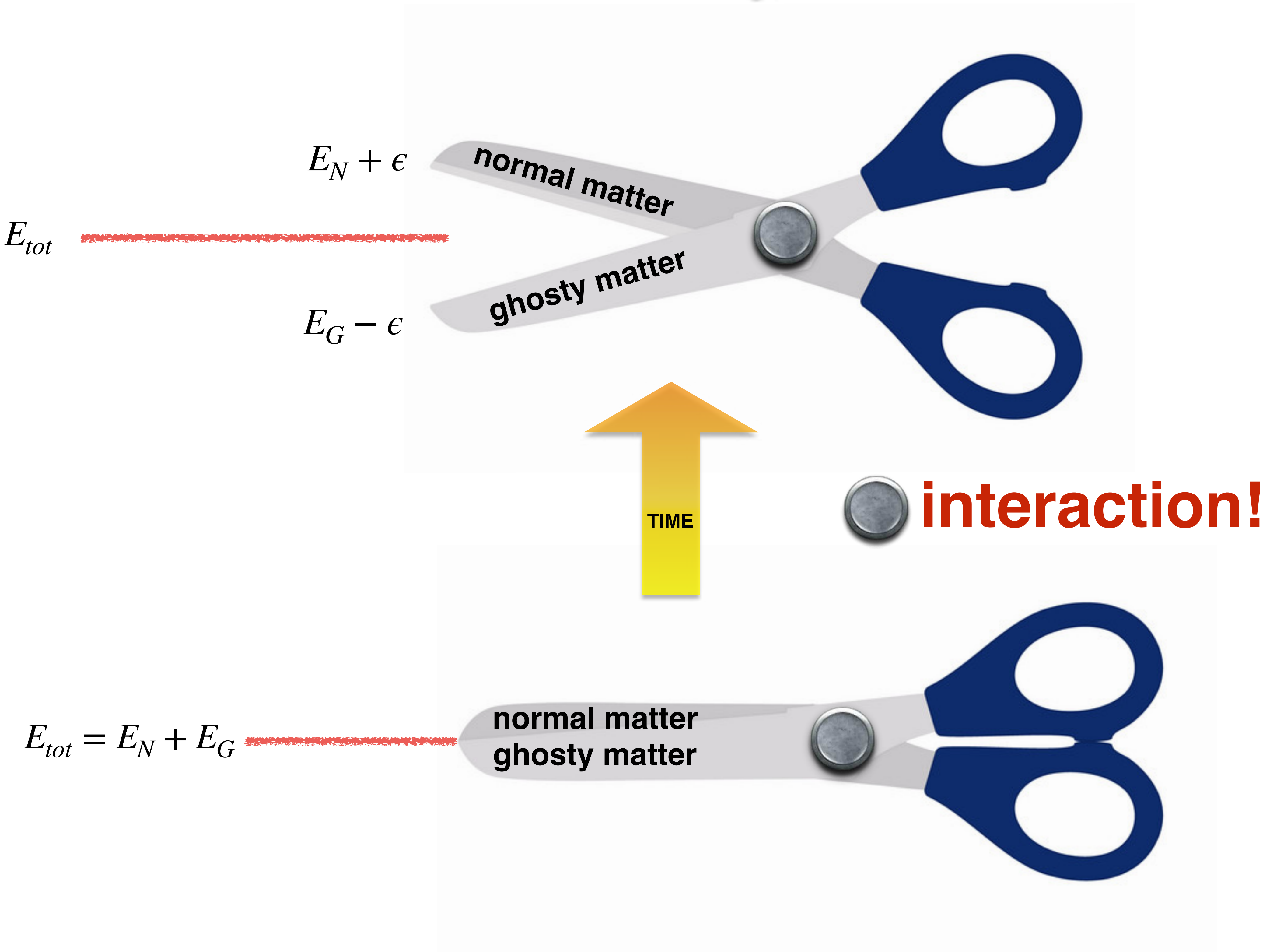

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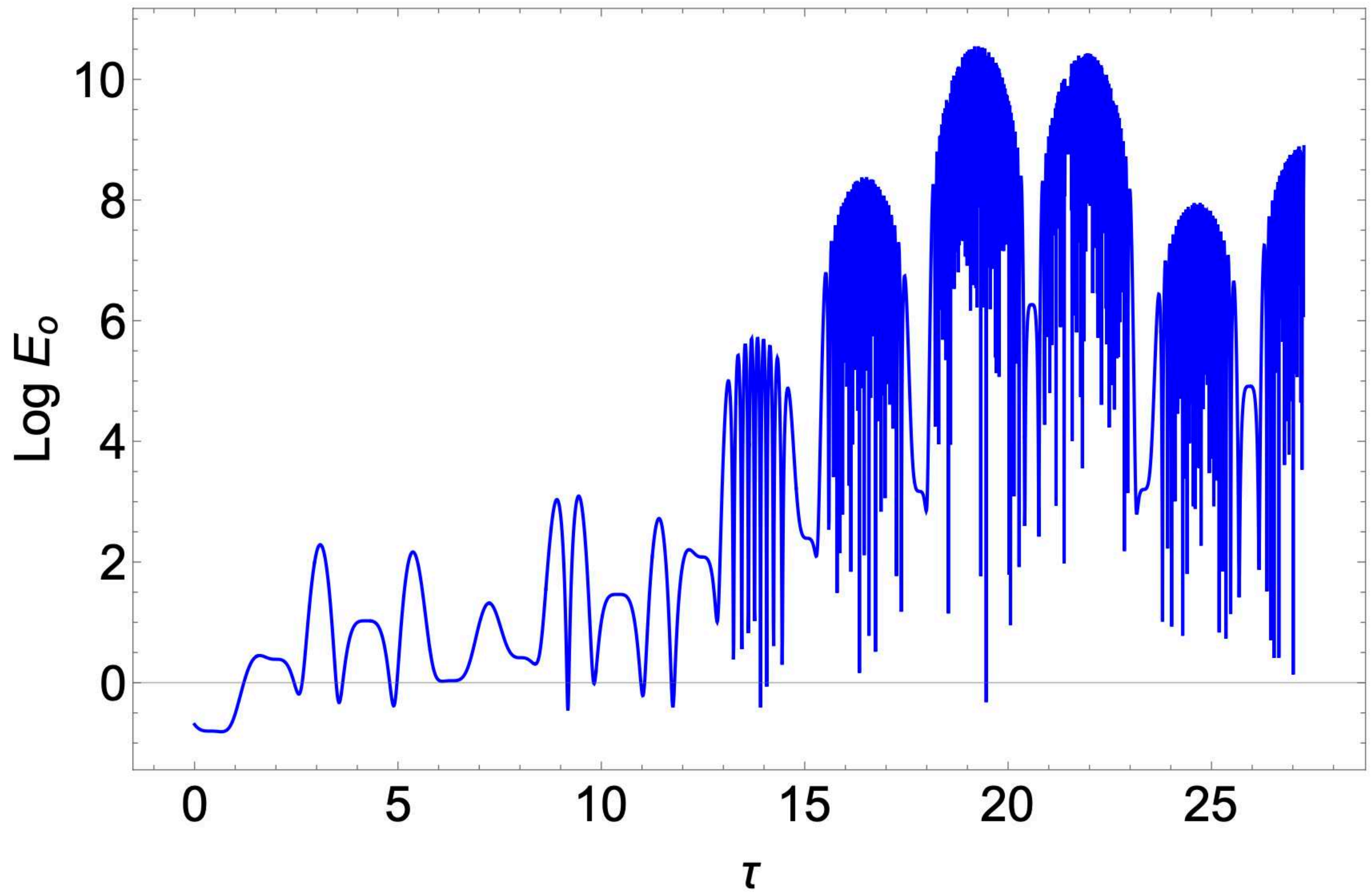

Figure 2: The growth of the logarithm of the energy of the observer is depicted for $\lambda = 4$, $\omega = 2.3$ and vacuum initial data (8) and (9).

How Unstable?

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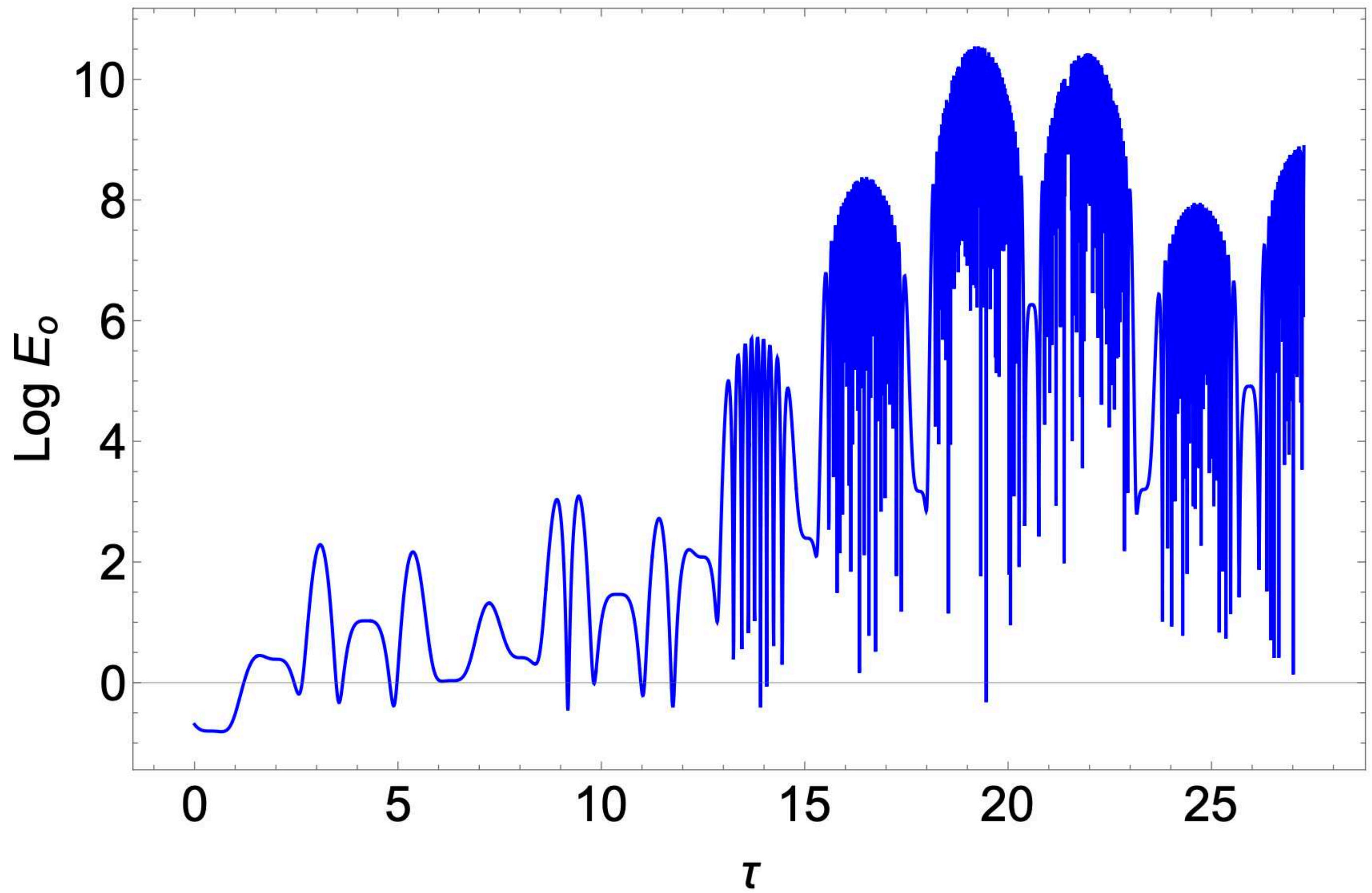

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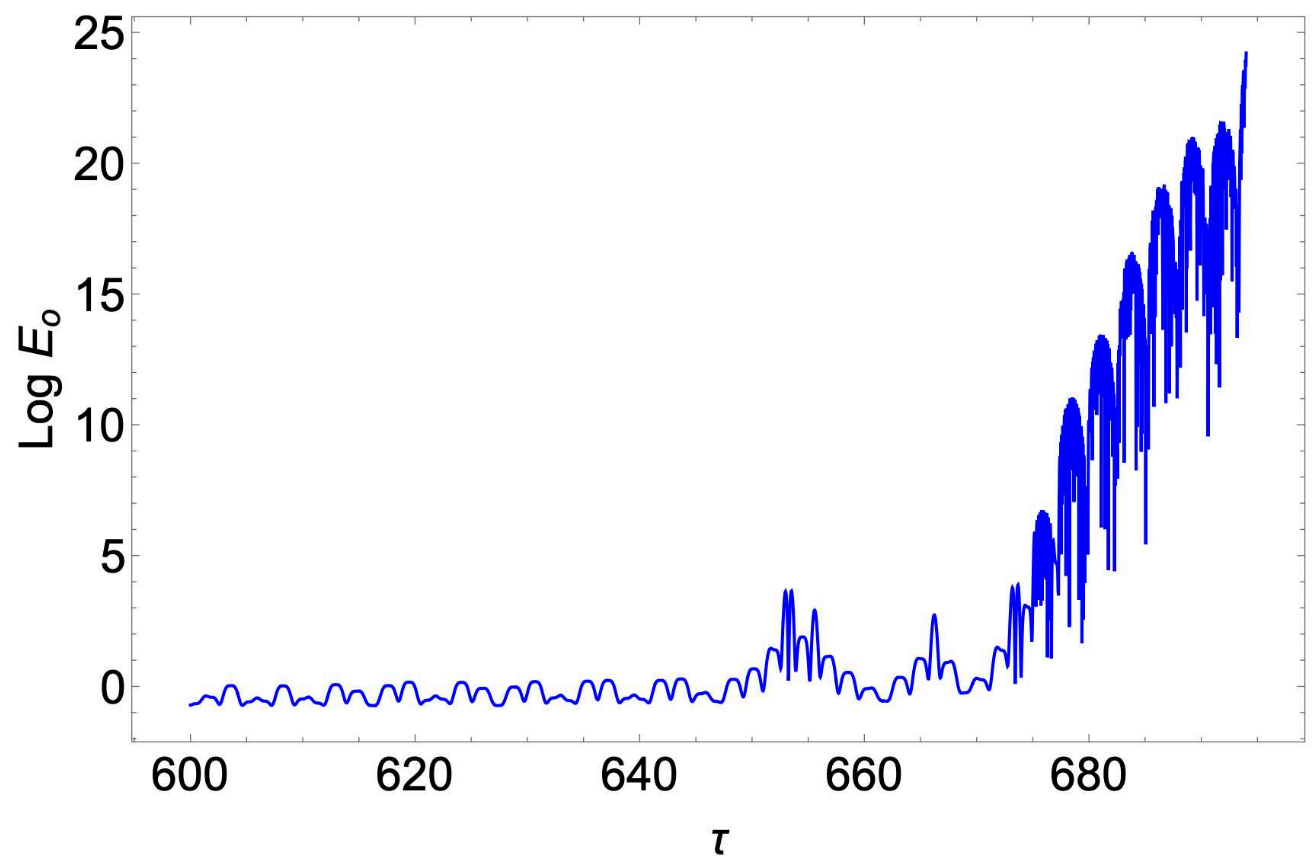

Figure 3: The growth of the logarithm of the energy of the observer is depicted for $\lambda = 2.35$, $\omega = 2.3$ and vacuum initial data (8) and (9). Here we see that the instability arises only much later after around a 100 of the periods of oscillation for the observer.

Our Stable PRL Model

Hamiltonian

$$H = \frac{1}{2}(p_x^2 + x^2) - \frac{1}{2}(p_y^2 + y^2) + V_I(x, y)$$

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Normal Oscillator | Ghosty Oscillator | Interaction Potential

$$V_I\left(x, y\right) = \frac{\lambda}{\sqrt{1 + 2\left(y^2 + x^2\right) + \left(y^2 - x^2\right)^2}}$$

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Interaction is bounded $0 < V_I(x, y)\,\lambda^{-1} \leq 1$

Potential

$$V_I(x,y) = \lambda\left[\left(x^2 - y^2 - 1\right)^2 + 4x^2\right]^{-1/2}$$

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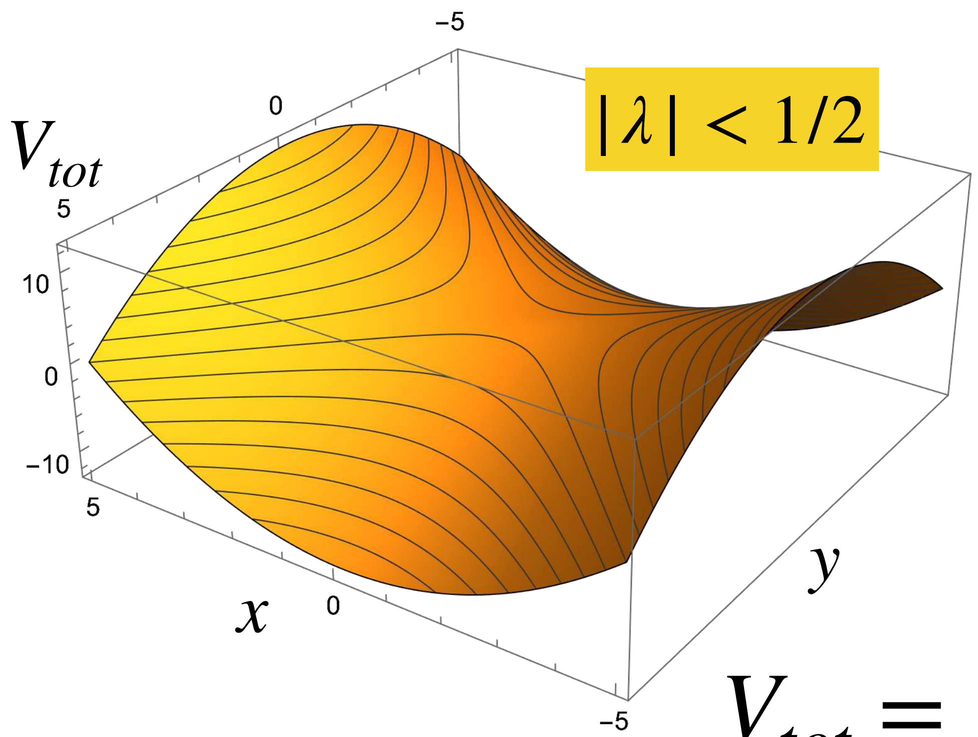

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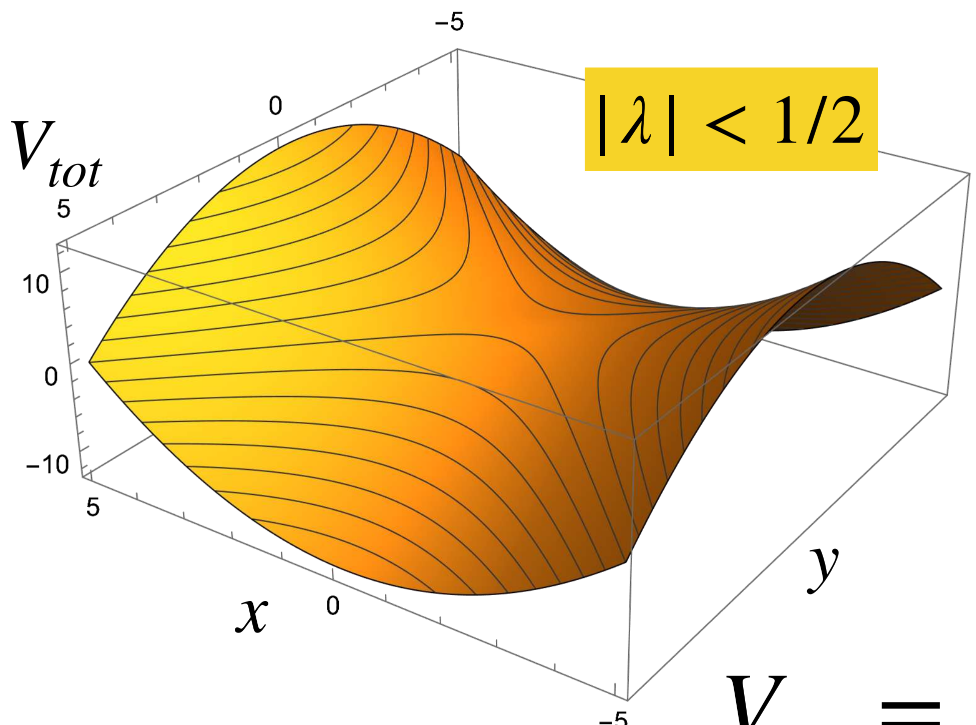

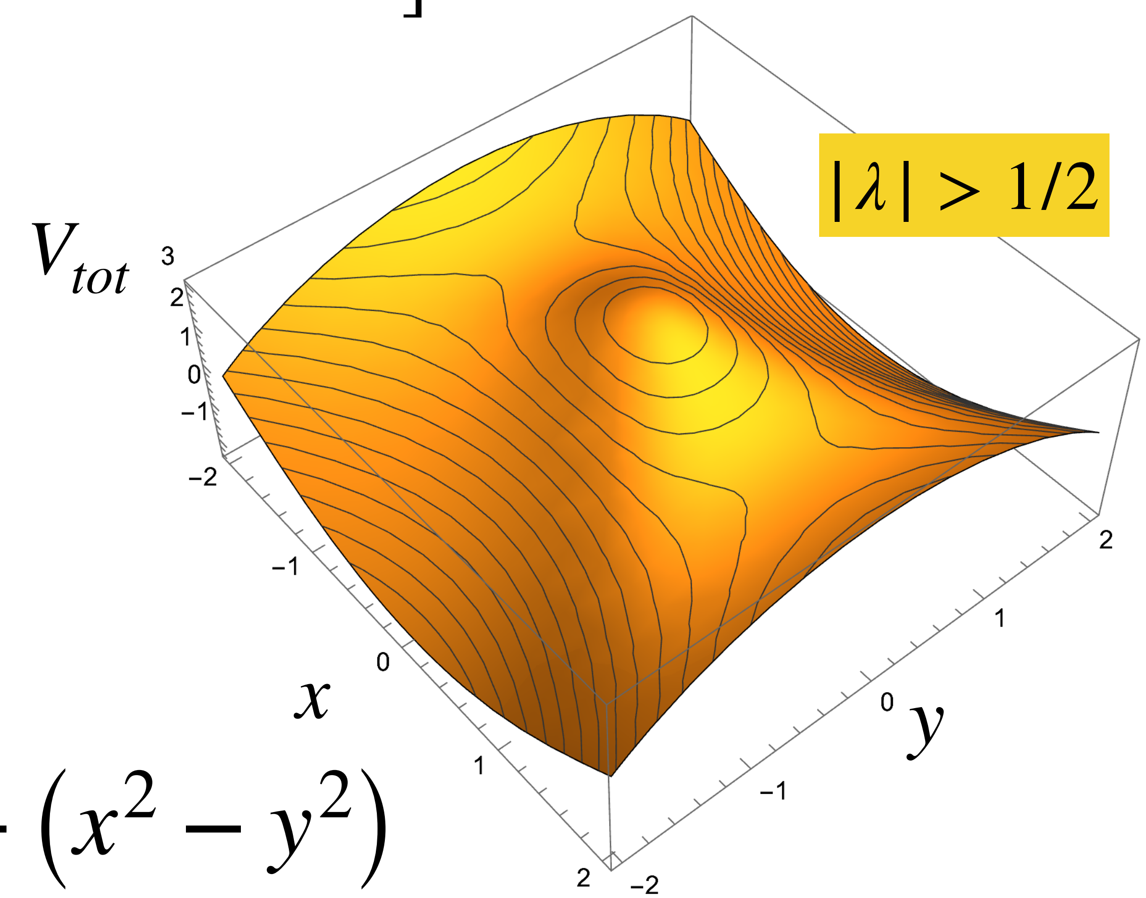

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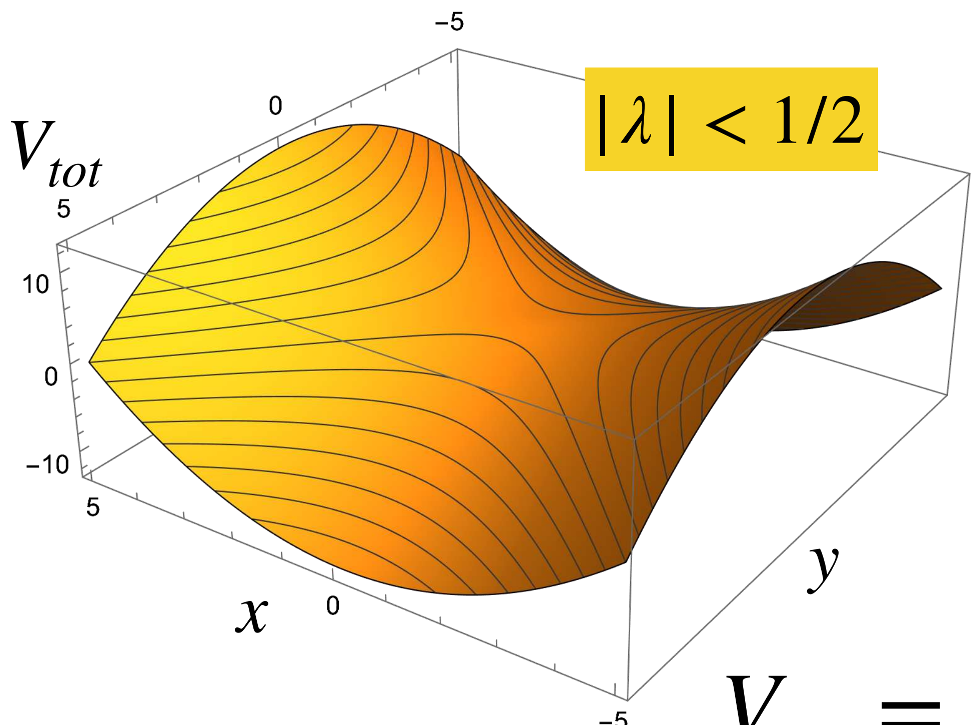

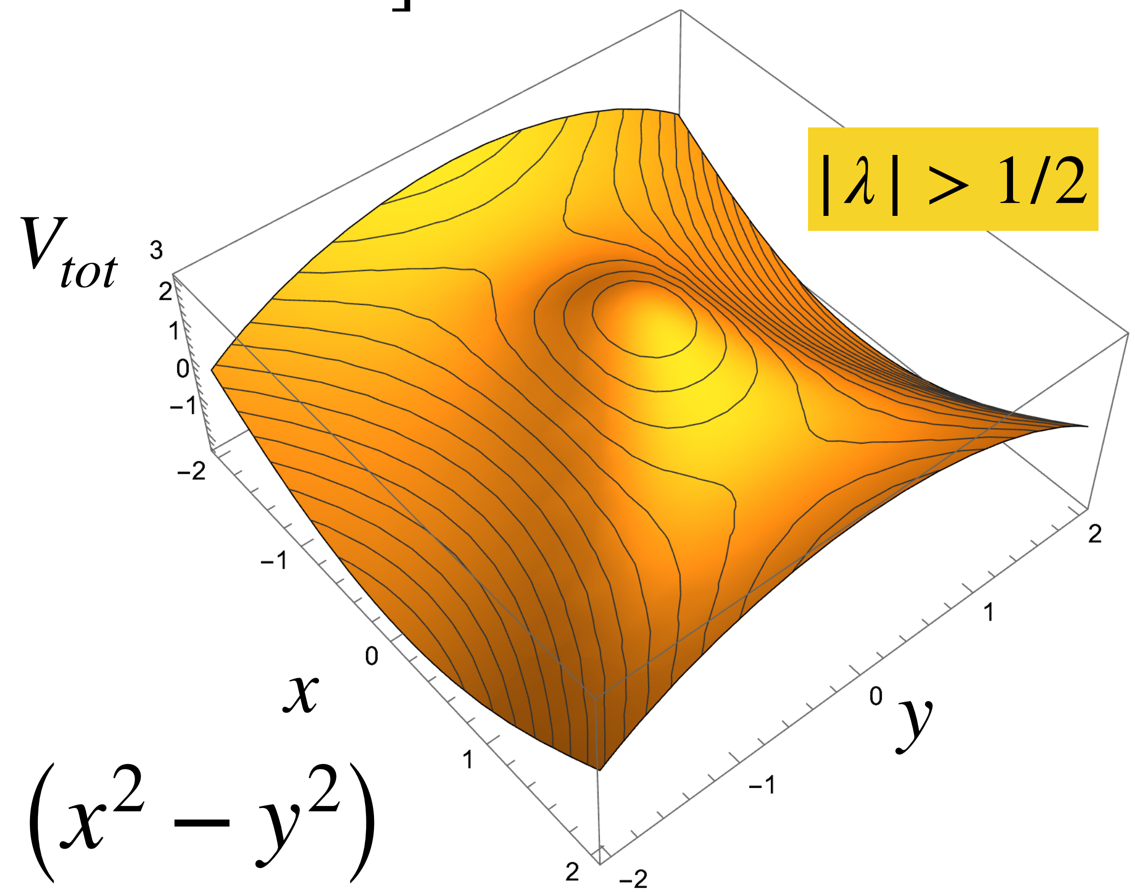

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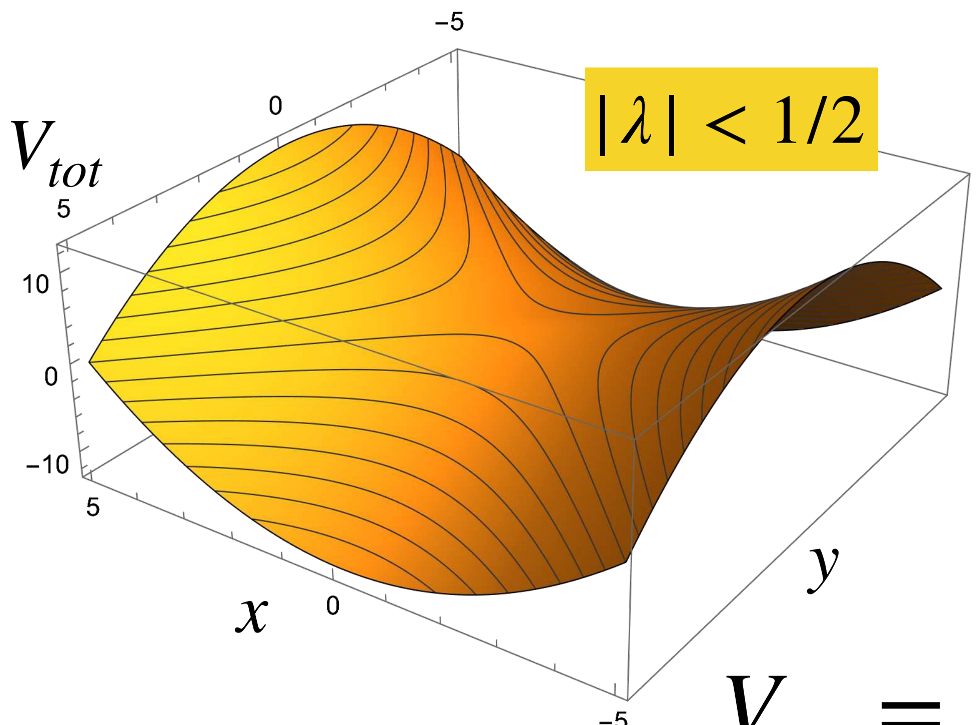

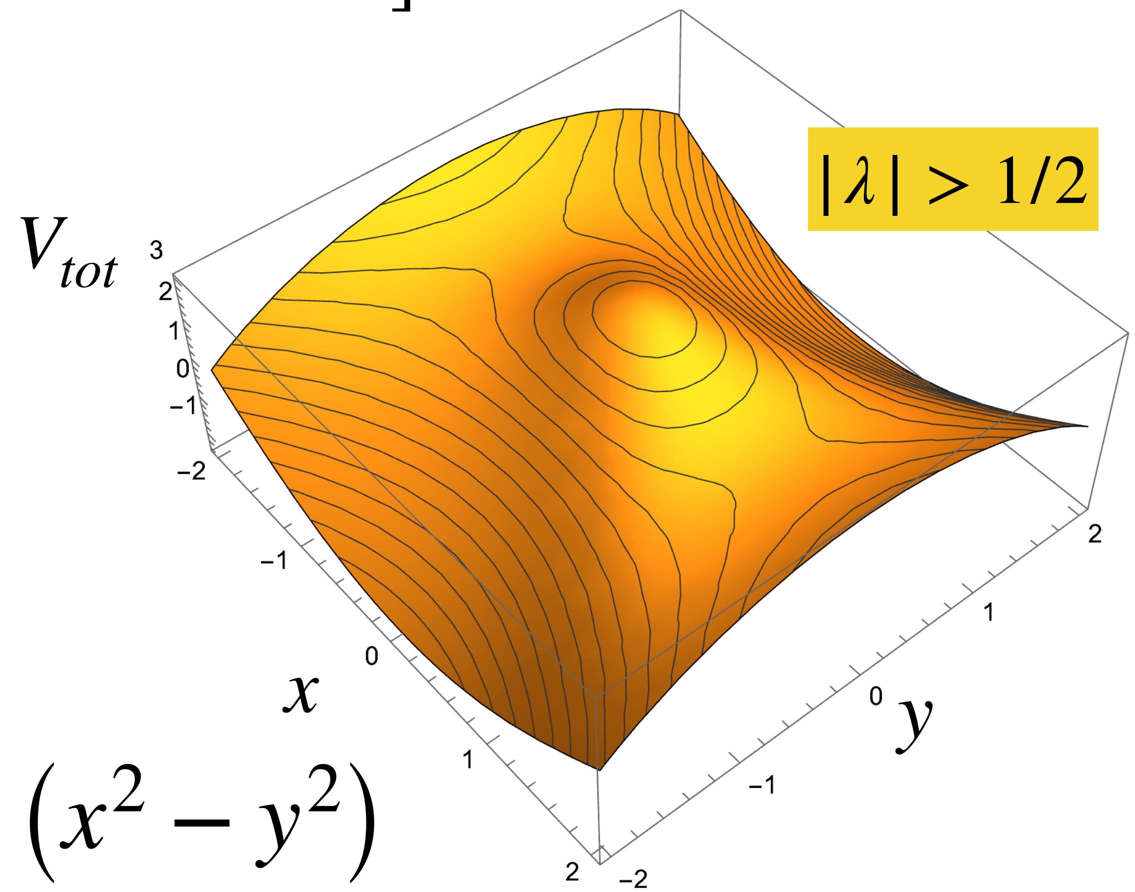

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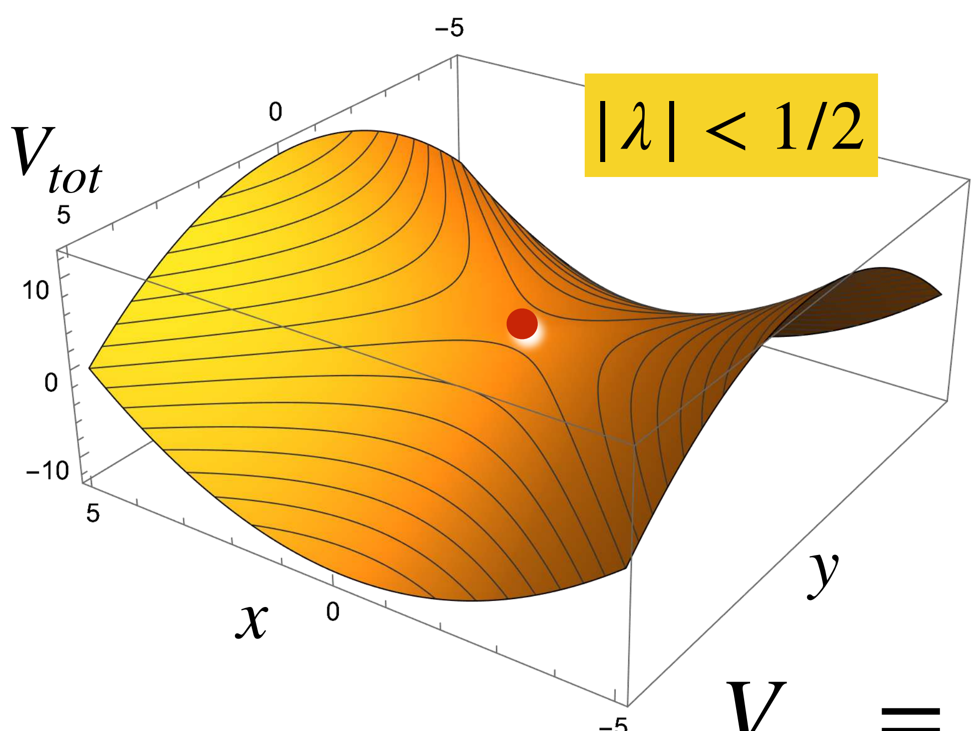

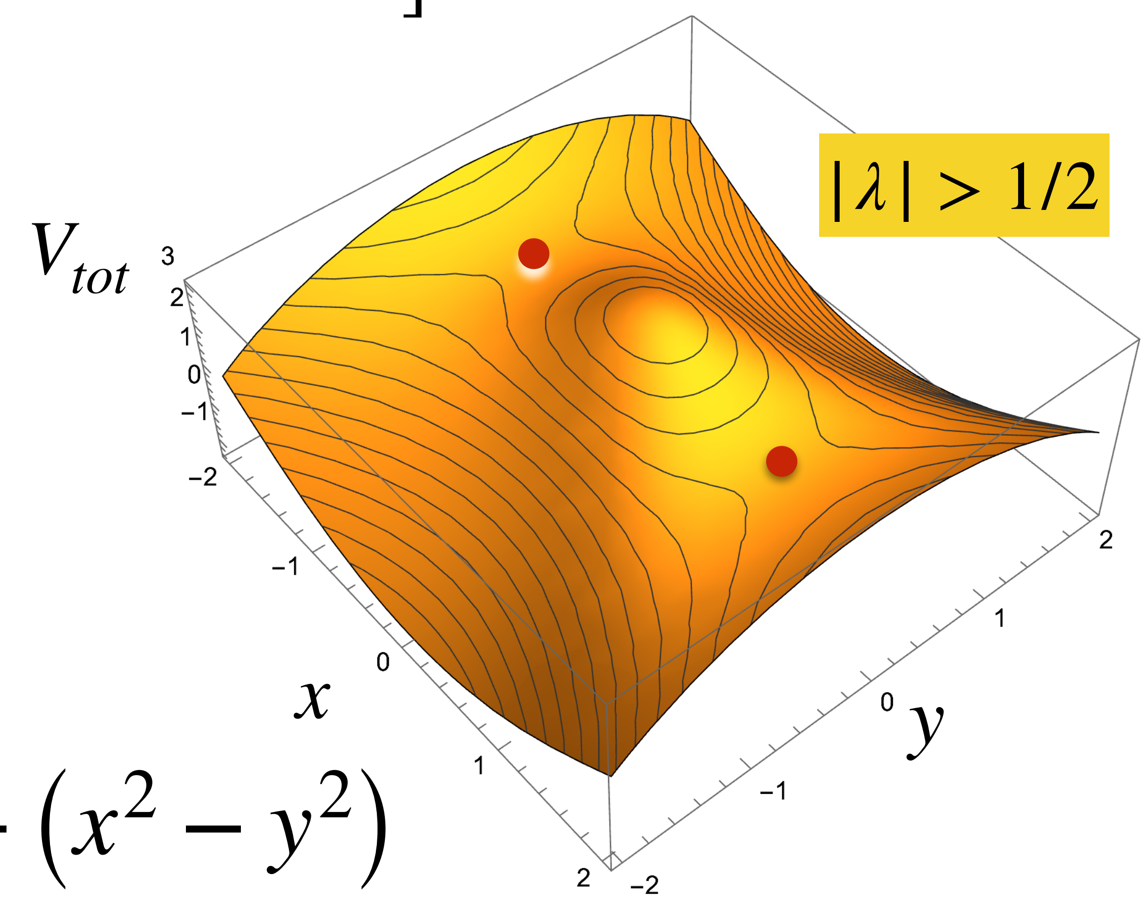

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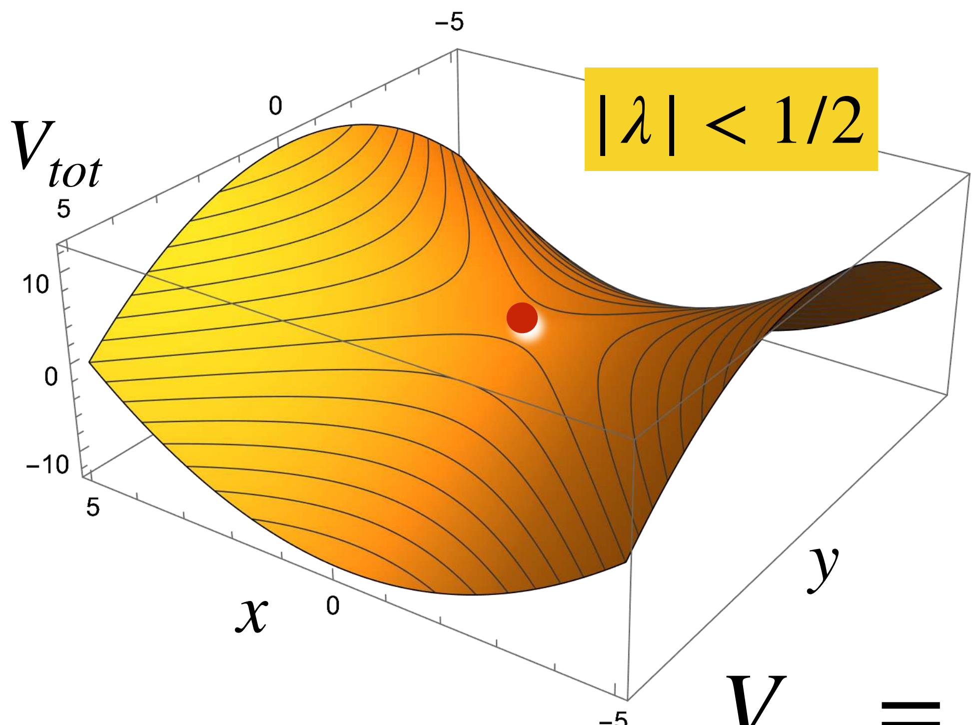

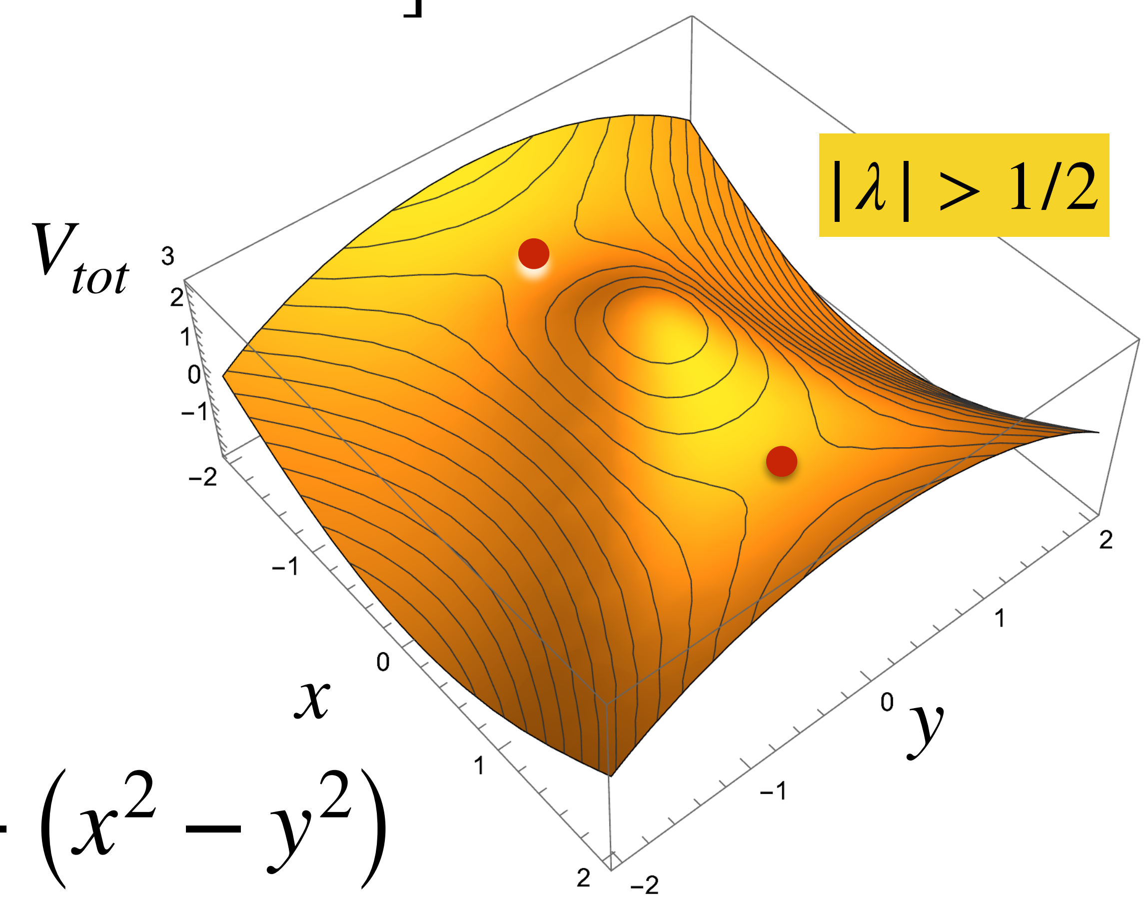

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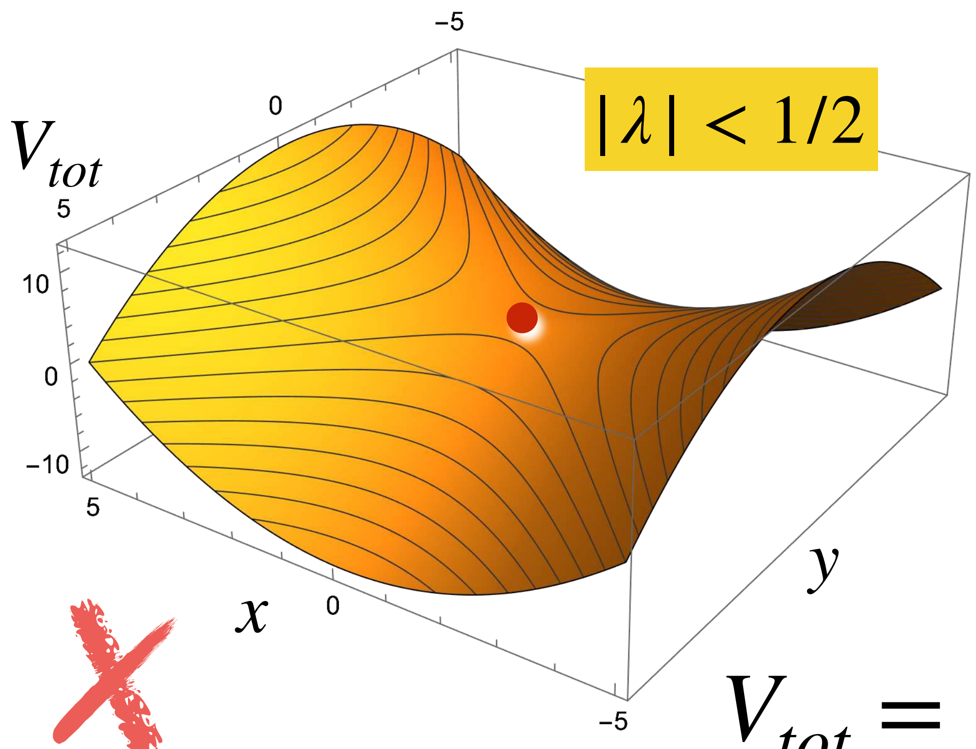

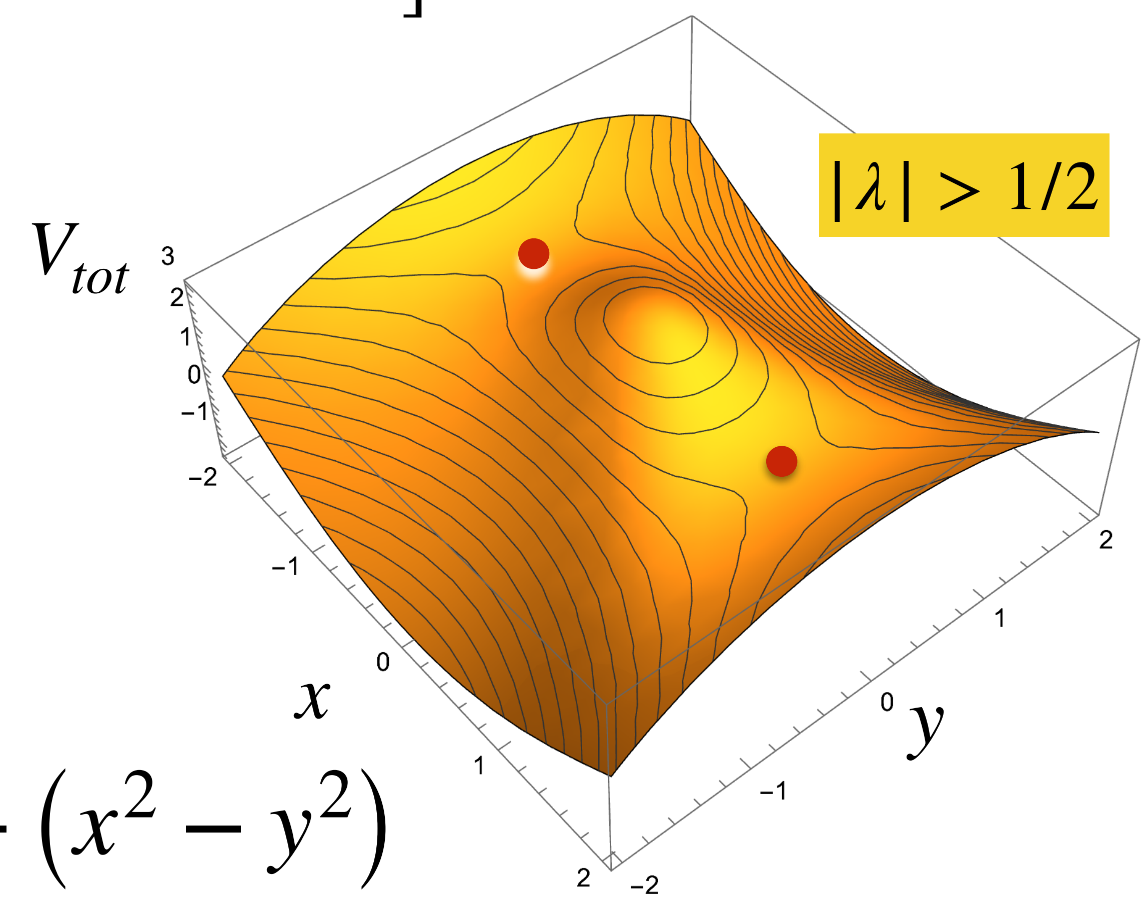

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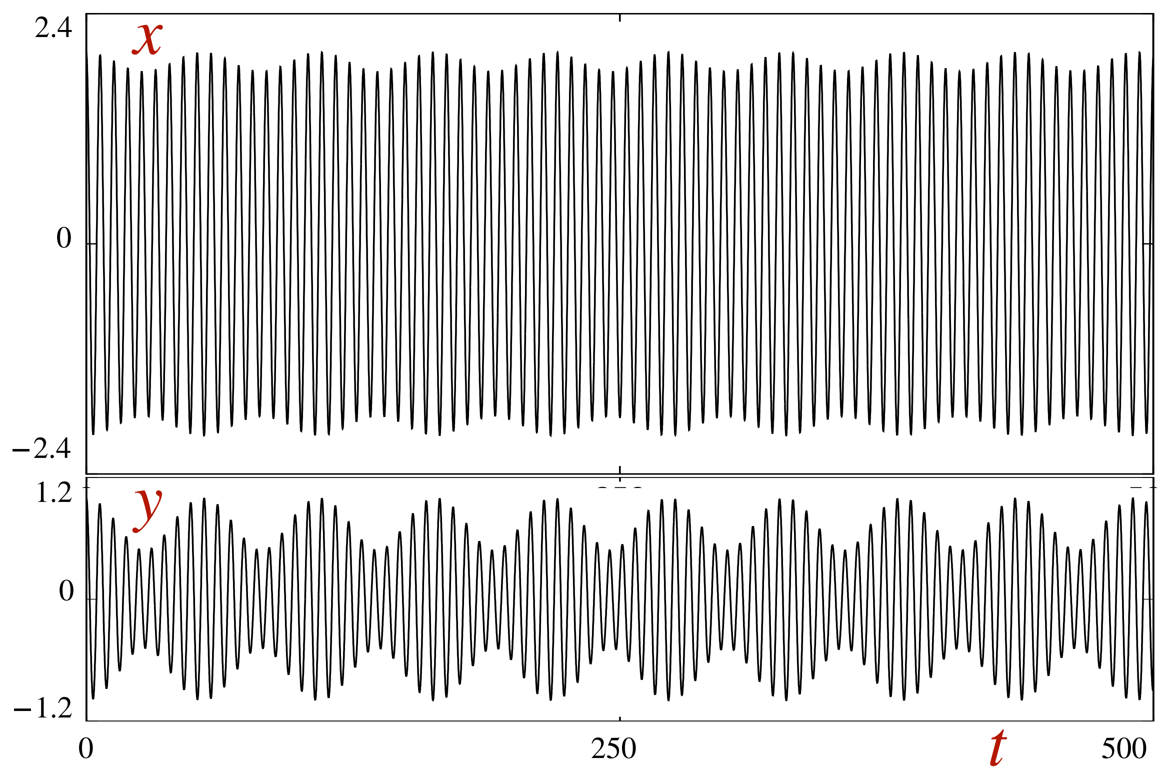

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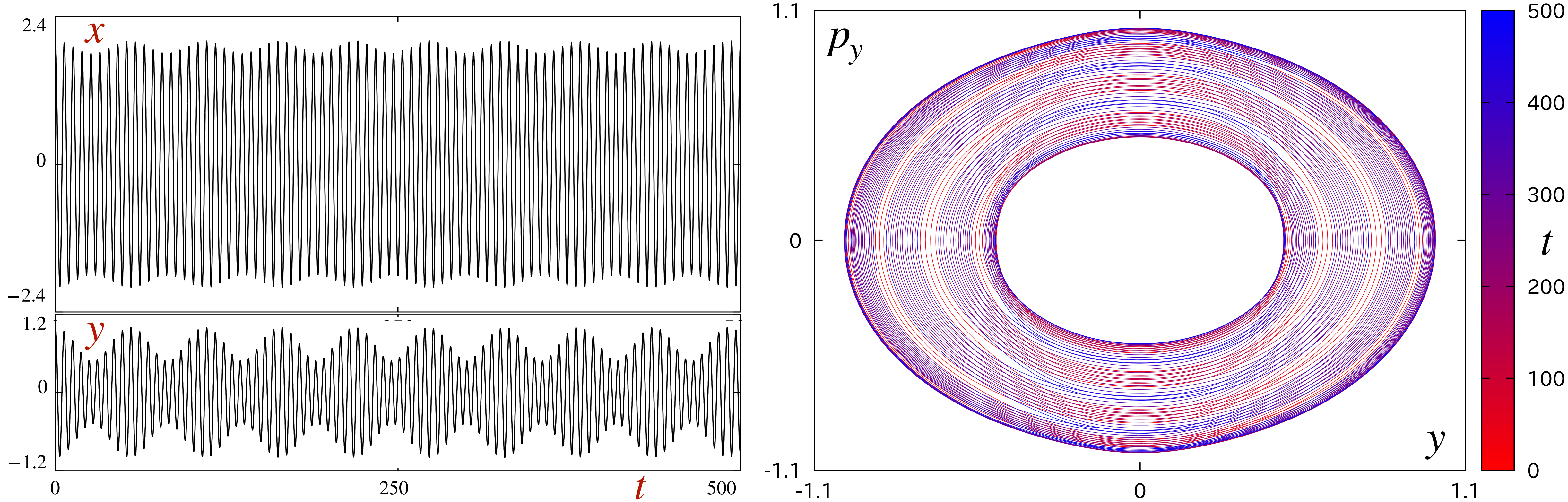

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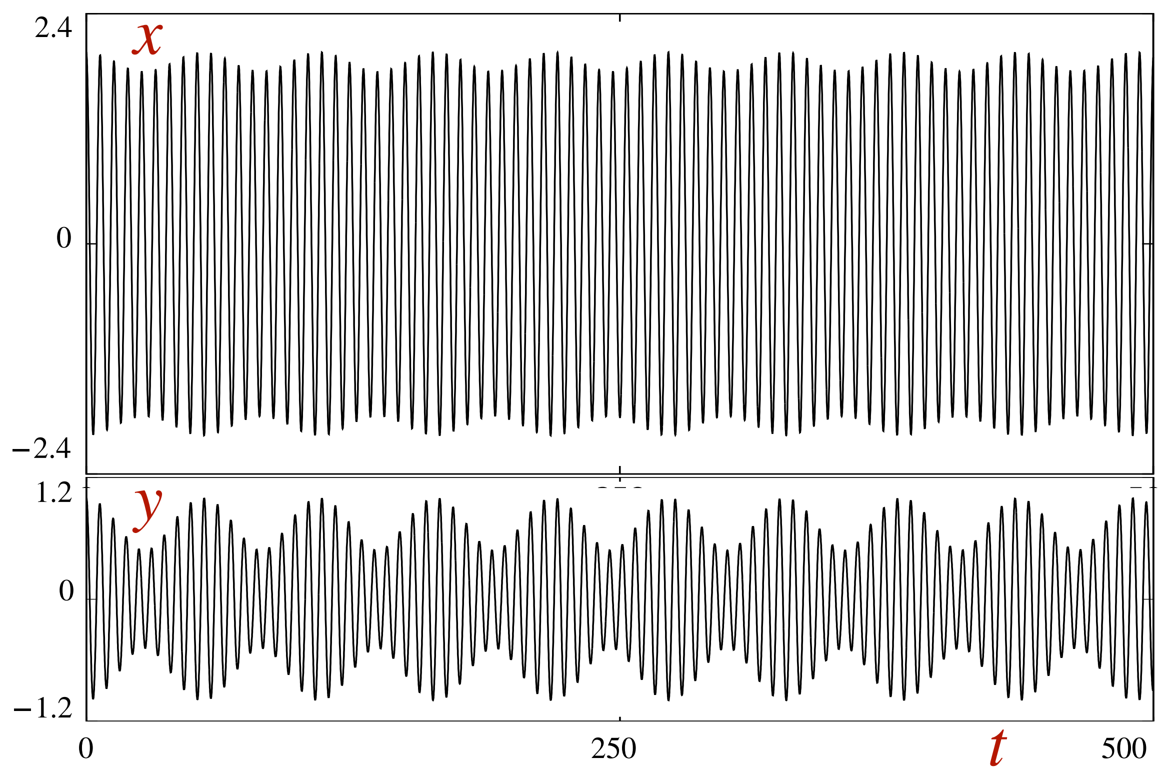

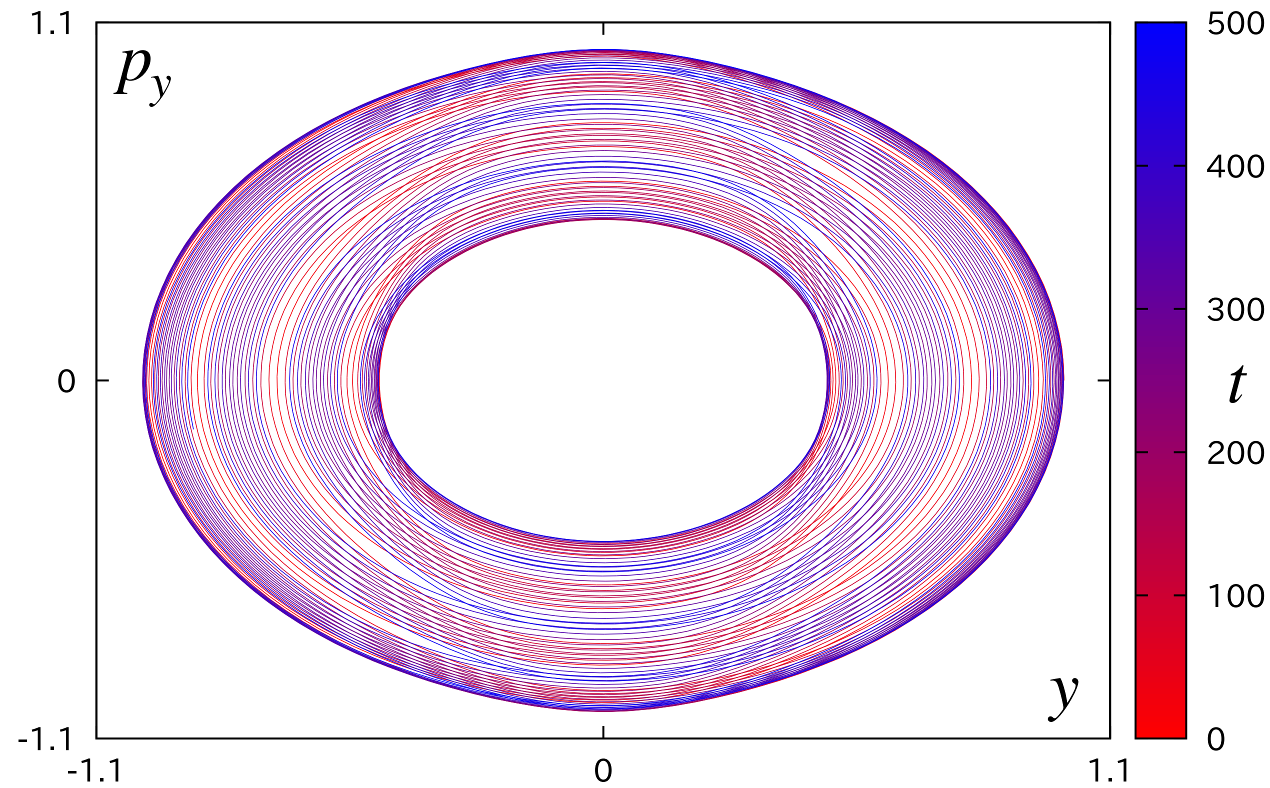

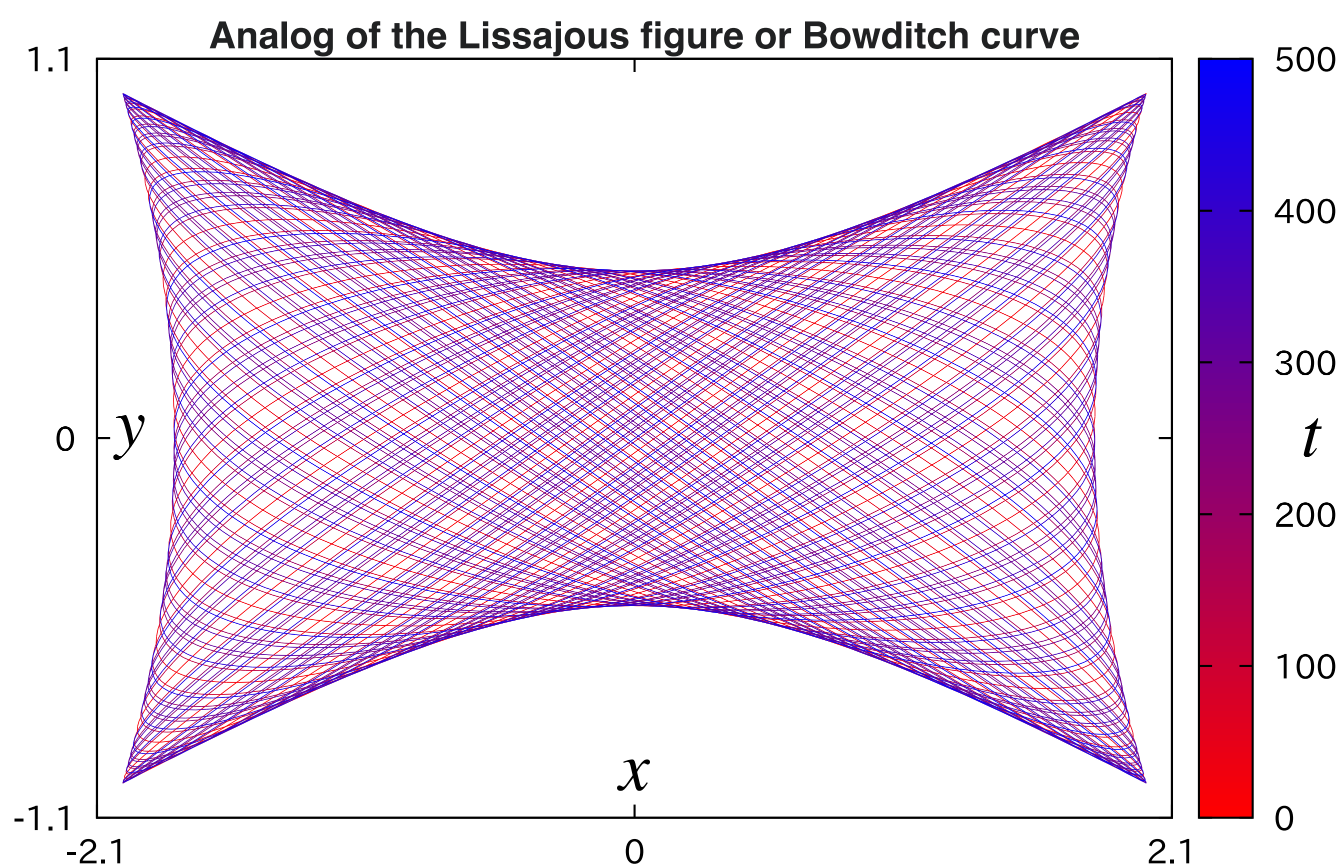

Why is it stable?

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What is the black magic?

First Integral and the Power of Imagination

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$$\mathscr{E} = K^2 + \frac{1}{2}\left(p_x^2 + x^2\right) + \frac{1}{2}(p_y^2 + y^2) + \left(y^2 - x^2\right) V_I\left(x, y\right)$$

generator for hyperbolic rotations $K = p_y x + p_x y$

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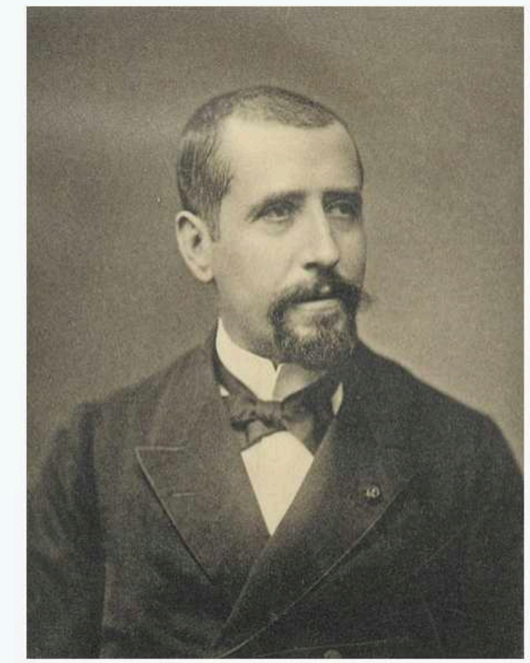

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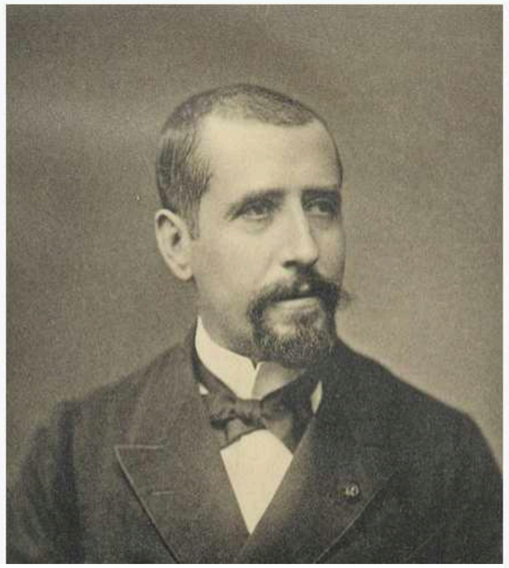

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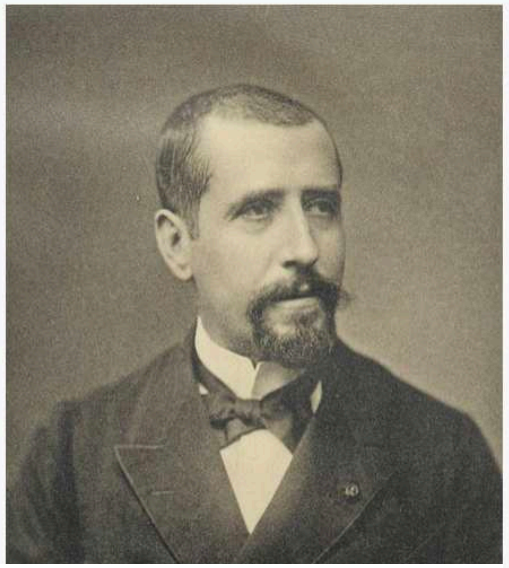

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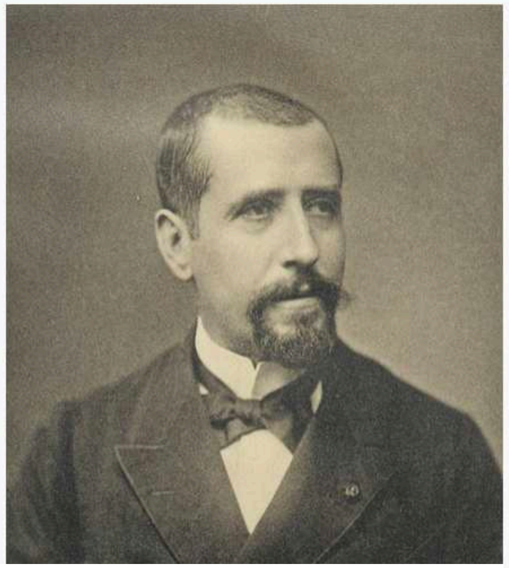

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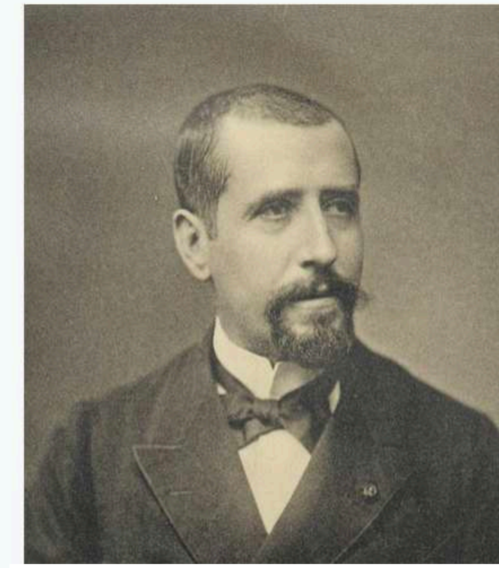

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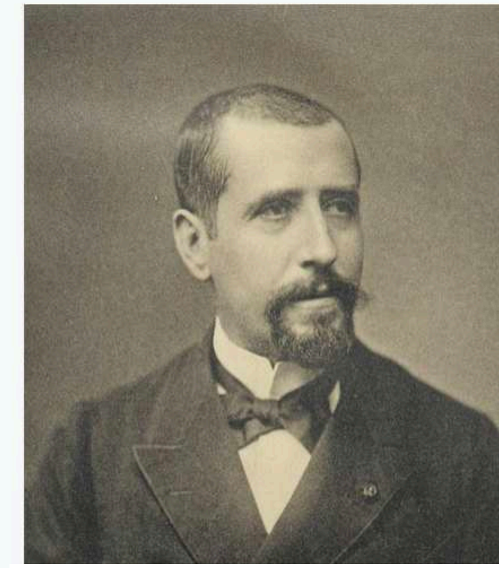

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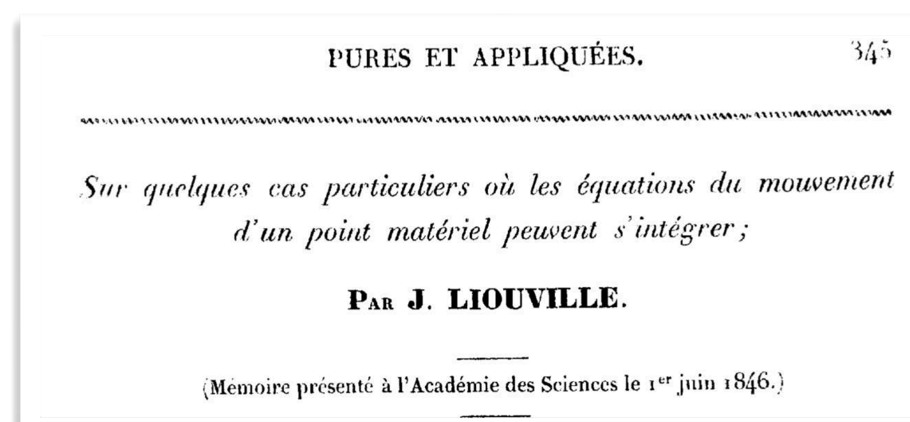

PURES ET APPLIQUÉES. 345

Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer;

PAR J. **LIOUVILLE**.

(Mémoire présenté à l'Académie des Sciences le 1er juin 1846.)

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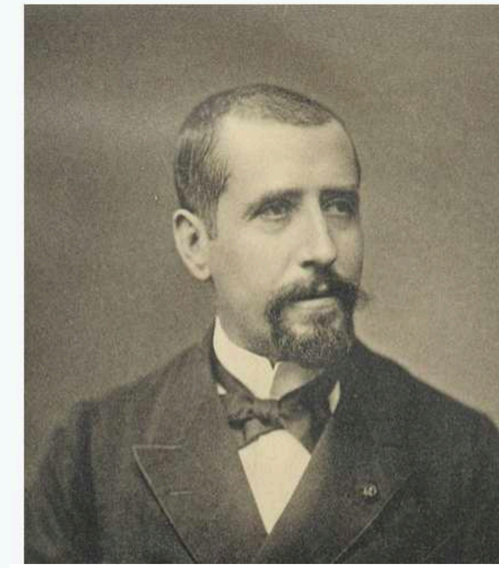

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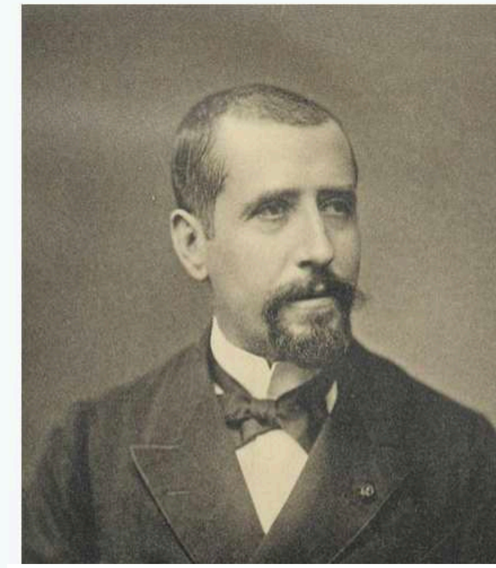
Is there any symmetry behind this conserved quantity $\mathscr{E}$?



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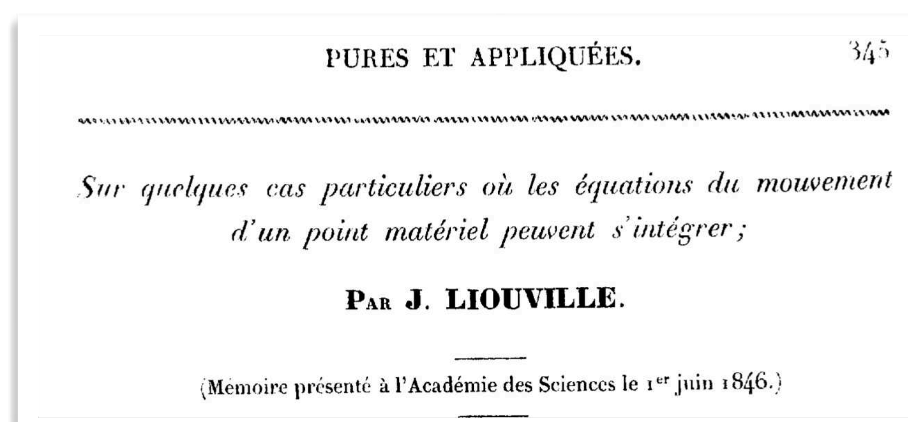

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the interaction part is always bounded

$$-|\lambda| \leq \left(y^2 - x^2\right) V_I\left(x, y\right) \leq |\lambda|$$

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⇓

for all times $\Sigma - |\lambda| \leq \mathscr{E} \leq \Sigma + |\lambda|$

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at initial point of time t_a $$\Sigma_a - |\lambda| \leq \mathscr{E} \leq \Sigma_a + |\lambda|$$

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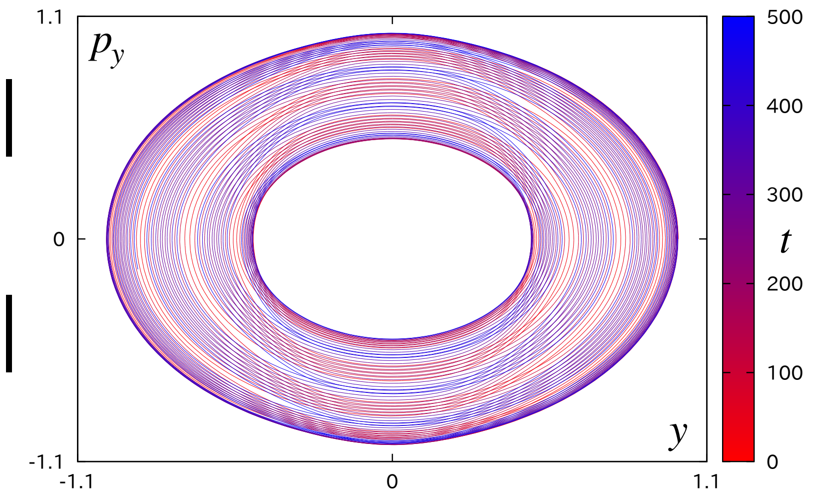

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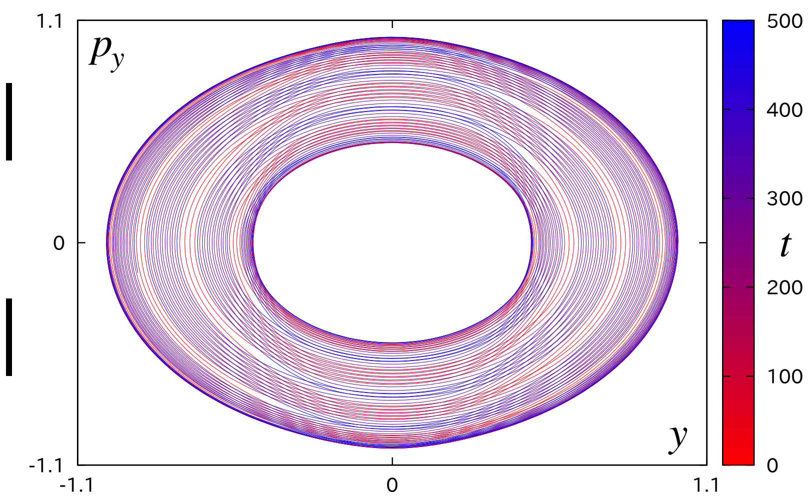

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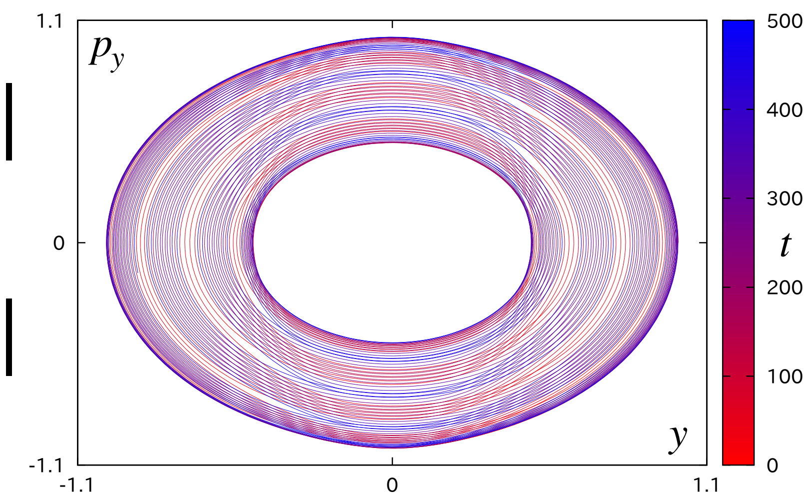
Thus the trajectory is confined in a stripe, as for $\xi = \left(x, y, p_x, p_y\right)$ we have $\left|\xi\right|^2 \leq 2\Sigma$

Finiteness of motion

at initial point of time t_a $\quad \Sigma_a - |\lambda| \le \mathscr{E} \le \Sigma_a + |\lambda|$

at any later point in time t_b $\quad \Sigma_b - |\lambda| \le \mathscr{E} \le \Sigma_b + |\lambda|$



Σ is positive definite and
is confined in a stripe

$$\Sigma_a - 2|\lambda| \le \Sigma_b \le \Sigma_a + 2|\lambda|$$

Thus the trajectory is confined in a stripe, as for $\xi = \left(x, y, p_x, p_y\right)$ we have $\left|\xi\right|^2 \le 2\Sigma$

System always evolves in a finite region of phase space

Lyapunov Stability

$$\mathscr{E} = \Sigma + \left(y^2 - x^2\right) V_I\left(x, y\right)$$

where

$$\Sigma = K^2 + \frac{1}{2}\left(p_x^2 + x^2\right) + \frac{1}{2}(p_y^2 + y^2)$$

Aleksandr Mikhailovich Lyapunov

The General Problem of the Stability of Motion, Doctoral dissertation, Kharkov U. 1892

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$\mathscr{E}$ is a Lyapunov function
so that the system is stable at the origin for $|\lambda| < 1/2$

Realisation through Higher Derivatives

Realisation through Higher Derivatives

$$L(q,\ddot{q}) = \left(\ddot{q}+q\right)\left(2p_2+(2p_2)^{-1}\right)$$

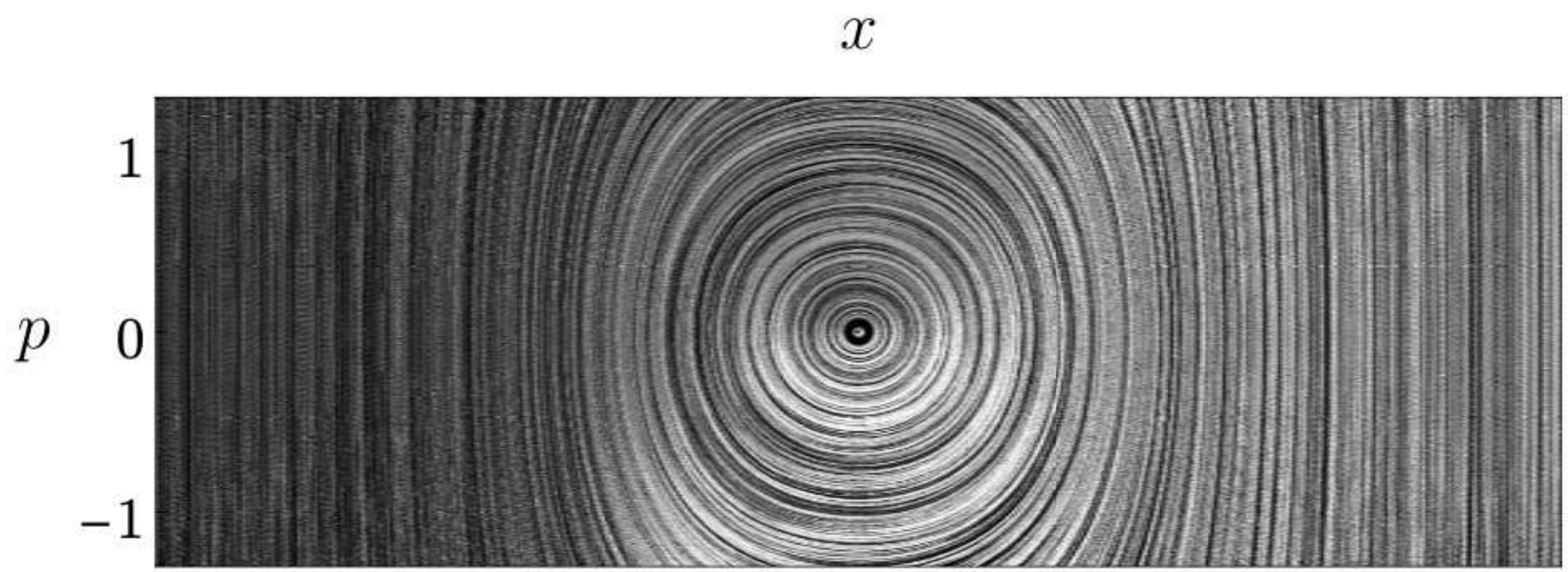
where $p_2 \equiv p_2\left(q,\ddot{q}\right)$ is the solution of

$$(\ddot{q}+q)\sqrt{2q^2+1} = -\,2\lambda p_2(2p_2^2+1)^{-3/2}$$

In this way $p_2 = \partial L/\partial \ddot{q}$

Does “imagination” matter for stability?

$$H = \frac{p^2}{2} + \frac{1}{4}\cosh x$$



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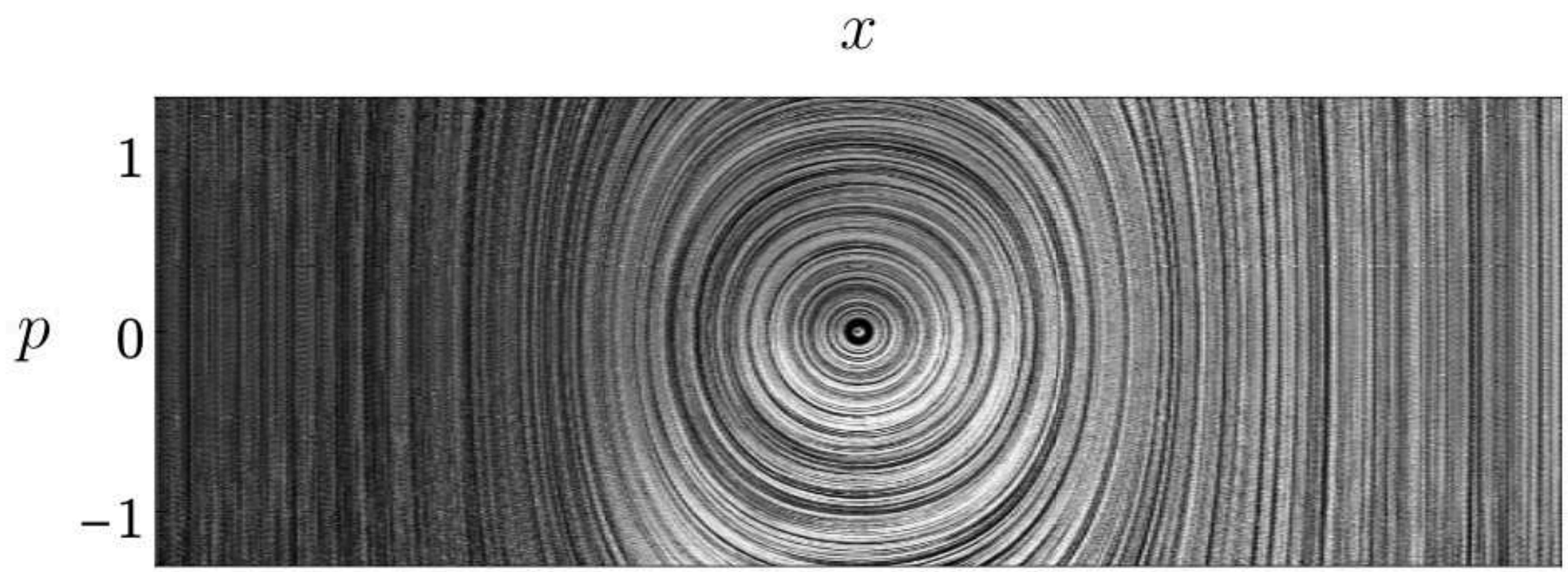

$$p = i\bar{p} \qquad x = -i\bar{x}$$

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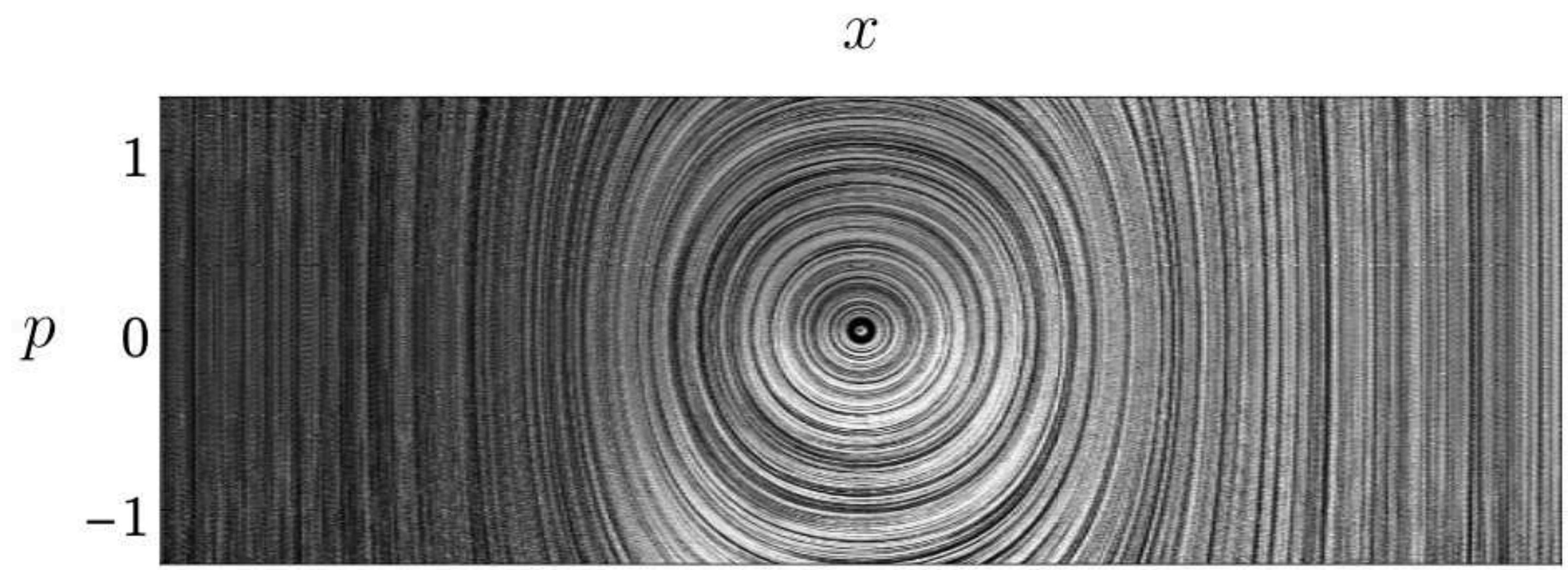

$$p = i\bar{p} \qquad x = -i\bar{x}$$

$$\bar{H} = -\frac{\bar{p}^2}{2} + \frac{1}{4}\cos\bar{x}$$

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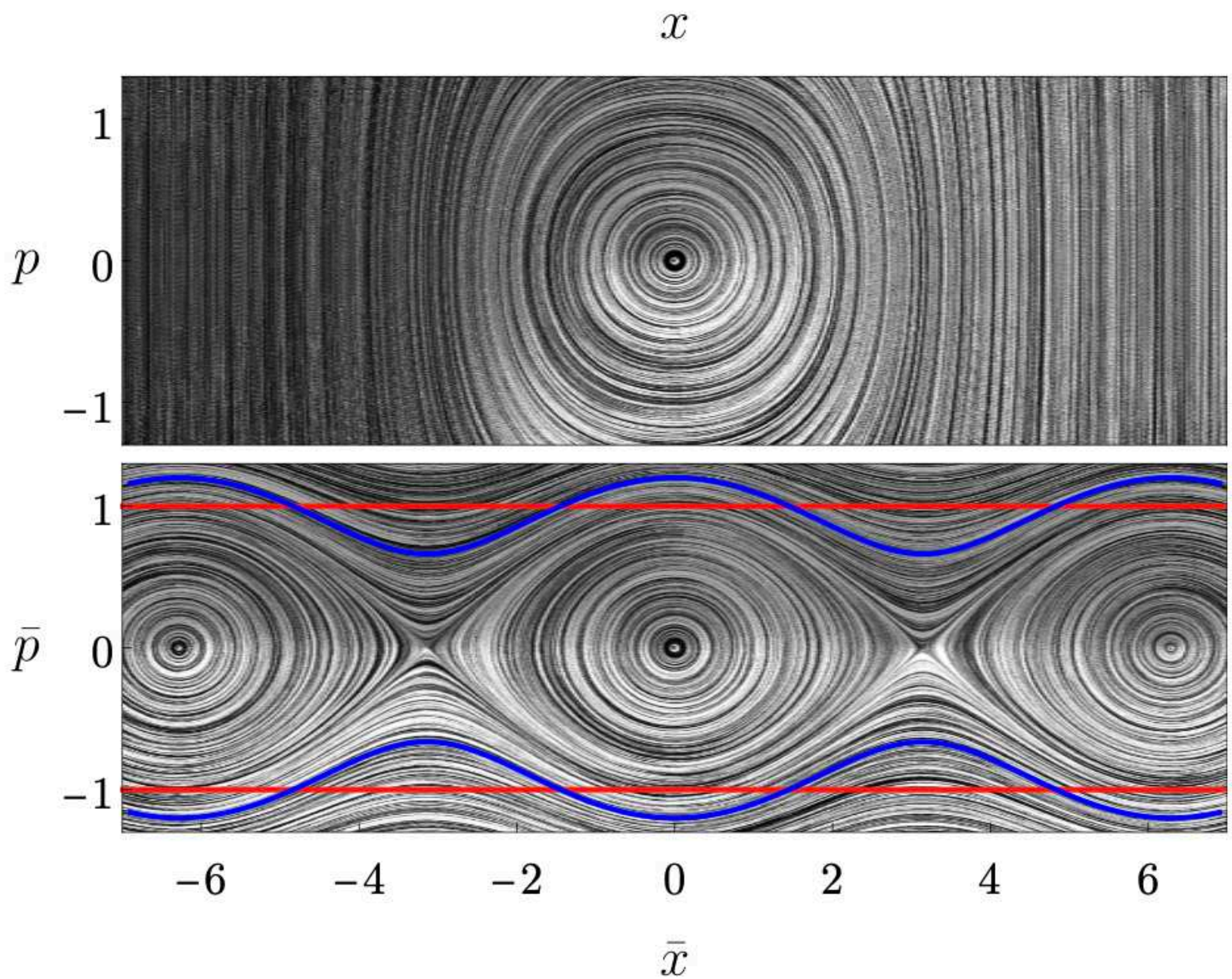

New large class of (Lagrange) stable ghosty systems

$$H_{LV} = \frac{p_x^2}{2} - \frac{p_y^2}{2} + V_{LV}(x, y)$$

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Condition for stability:

- $c > 0$
- $f(u)$ and $g(v)$ are bounded from below
- $f(u) \geqslant F_0\,|\,u\,|^{\zeta} > 0$
  $g(v) \geqslant G_0\,|\,v\,|^{\eta} > 0$
  with $\zeta > 2$ and $\eta > 2$

Dirichlet-Lagrange Theorem

Johann Peter Gustav
Lejeune Dirichlet

Giuseppe Ludovico
De la Grange Tournier

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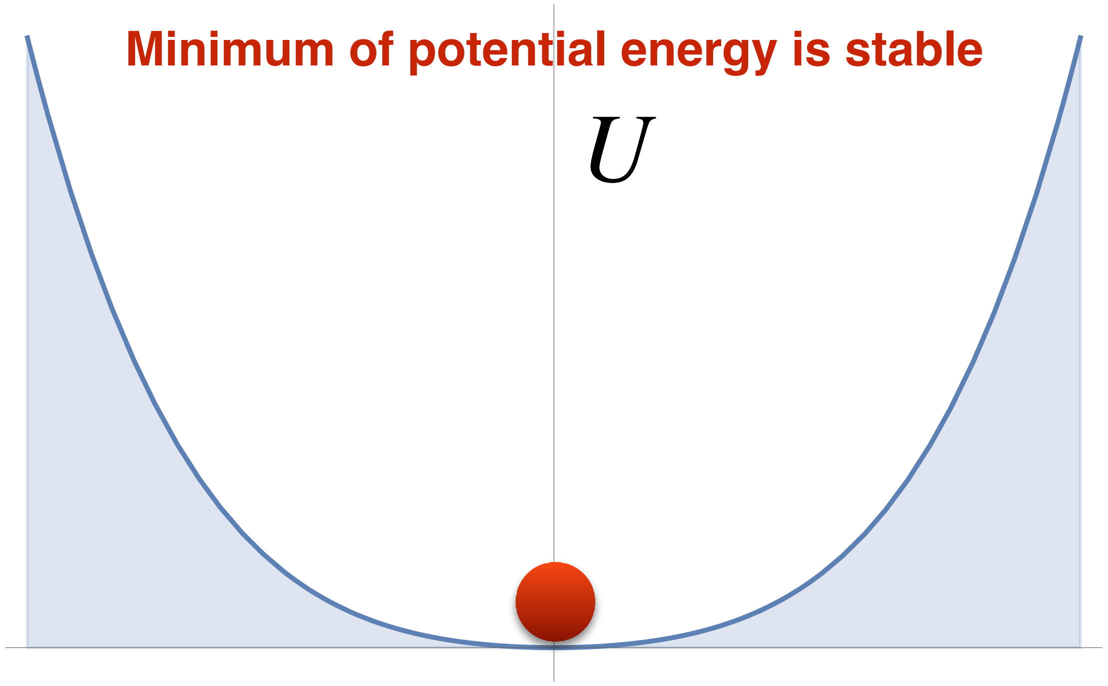

What is the black magic?

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Another First Integral!

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Another First Integral!

$$J_{LV} = \left(xp_y + yp_x\right)^2 + \frac{c}{2}\left(p_x^2 + p_y^2\right) + \mathcal{U}$$

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$$\mathcal{U}(u, v) = \frac{\left(2\,u^2 + c\right) g(v) + \left(2\,v^2 - c\right) f(u)}{u^2 + v^2}$$

What is the black magic?

Another First Integral!

"Energy"

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Another First Integral!

"Energy" "Positive Definite Kinetic Energy" "Potential Energy"

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Stable ghost with Polynomial Interaction

general polynomial potential

$$V_{LV}^{(N)} = \sum_{n=1}^{N} \frac{\mathscr{C}_n}{u^2 + v^2} \left[\left(u^2\right)^n - \left(-v^2\right)^n \right]$$

Stable ghost with Polynomial Interaction

constants

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minimal order polynomial potential with stable motion $N = 4$:

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"potential energy" of the first integral:

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Stable ghost with Polynomial Interaction

constants

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for Lagrange stability: $\mathscr{C}_4 > 0$

"Potential Energy" $\mathscr{U}_{LV}^{(4)}(x, y)$ and Vacua

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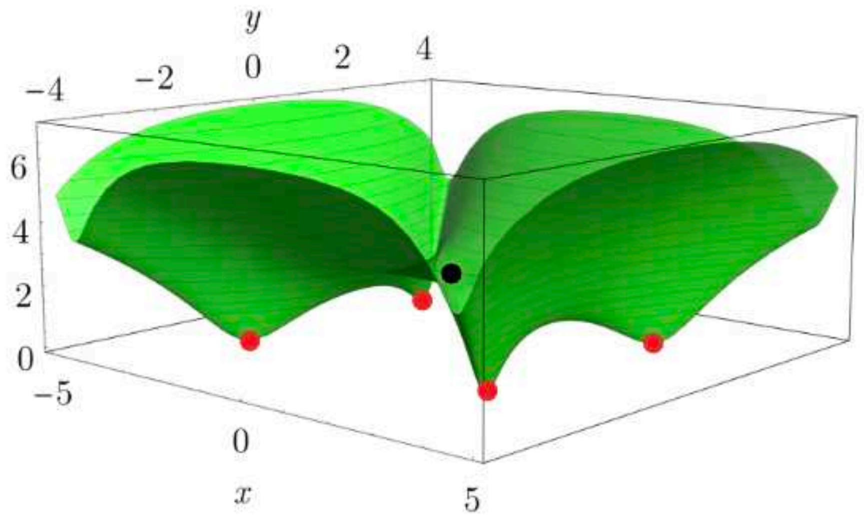

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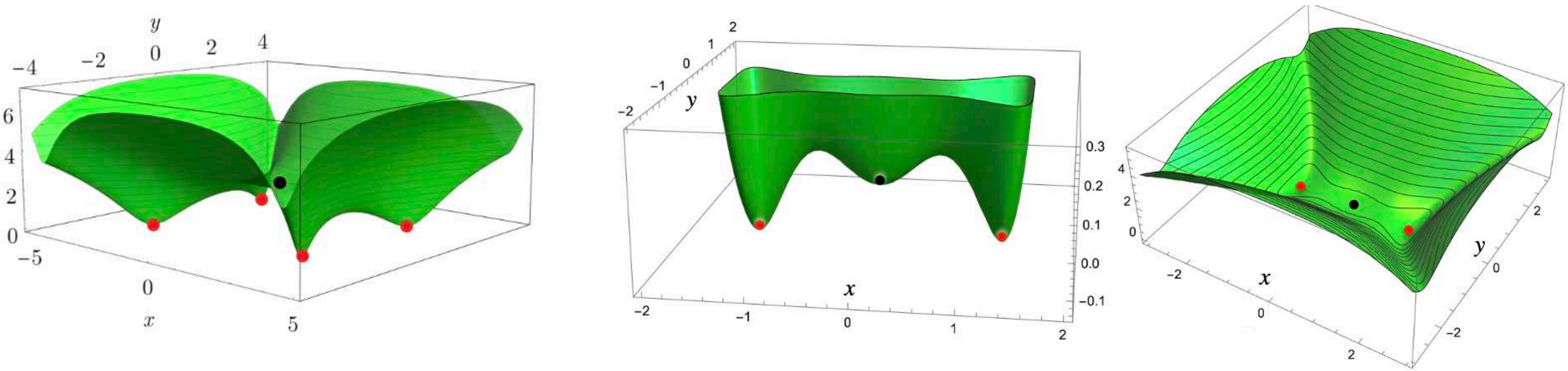

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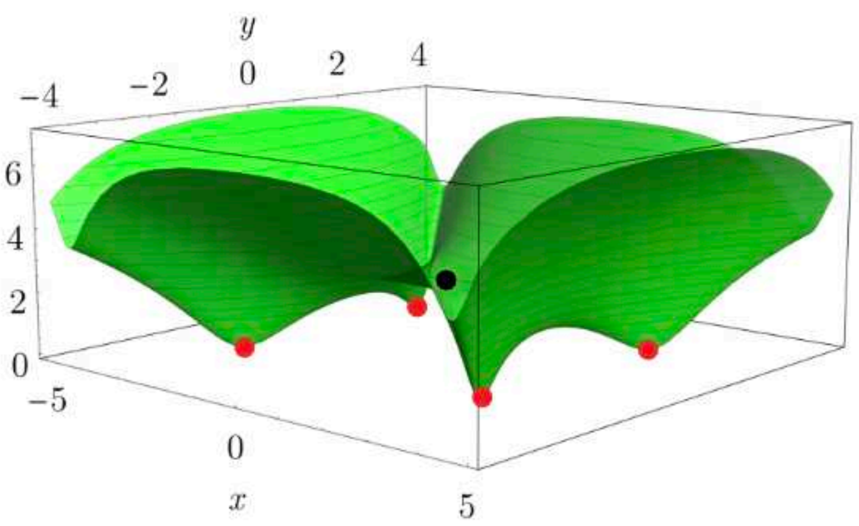

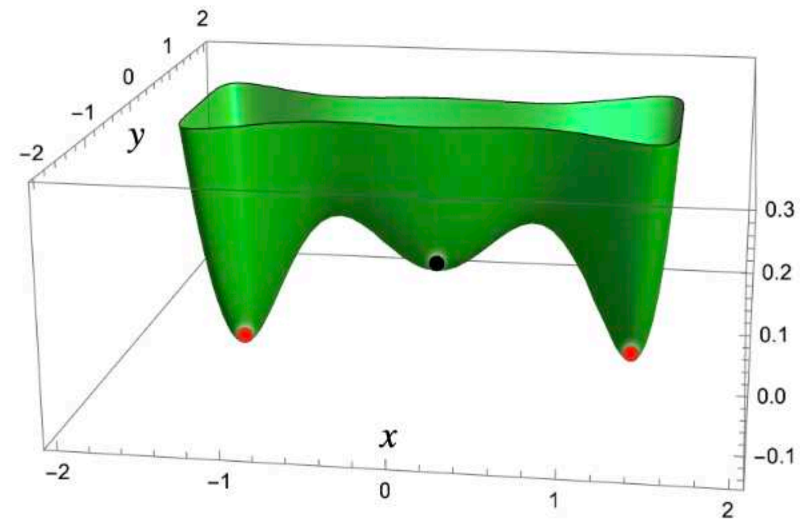

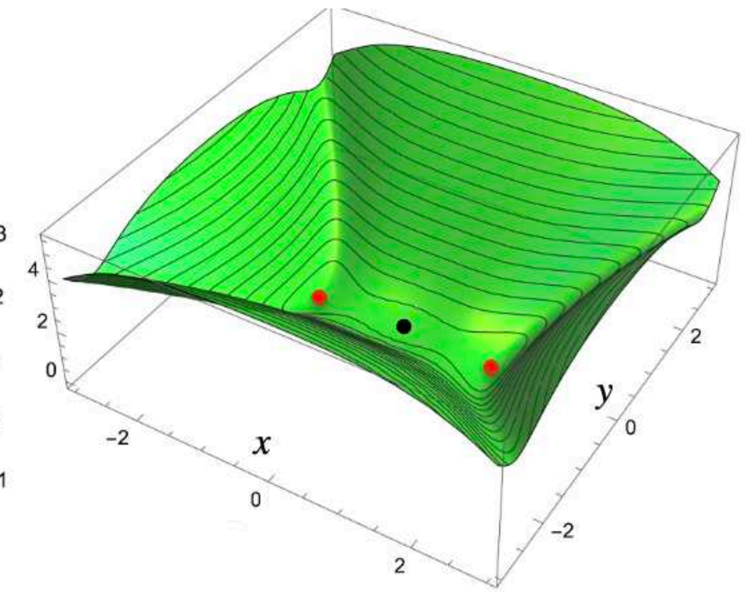

$$\Omega_x^2 = \frac{\omega_x^2}{2\mathscr{C}_4 c^2}$$

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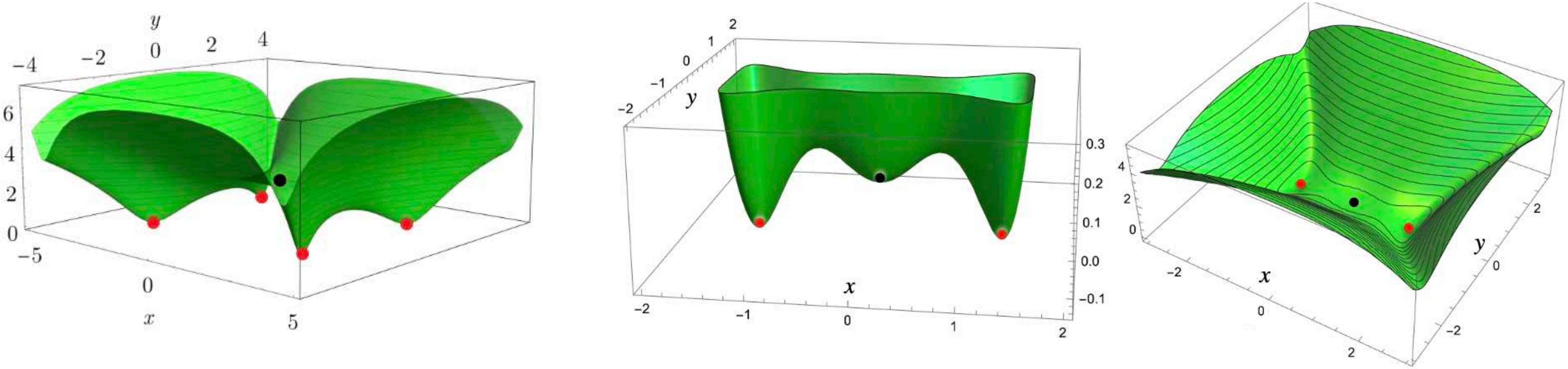
"Potential Energy" $\mathscr{U}_{LV}^{(4)}(x, y)$ and Vacua



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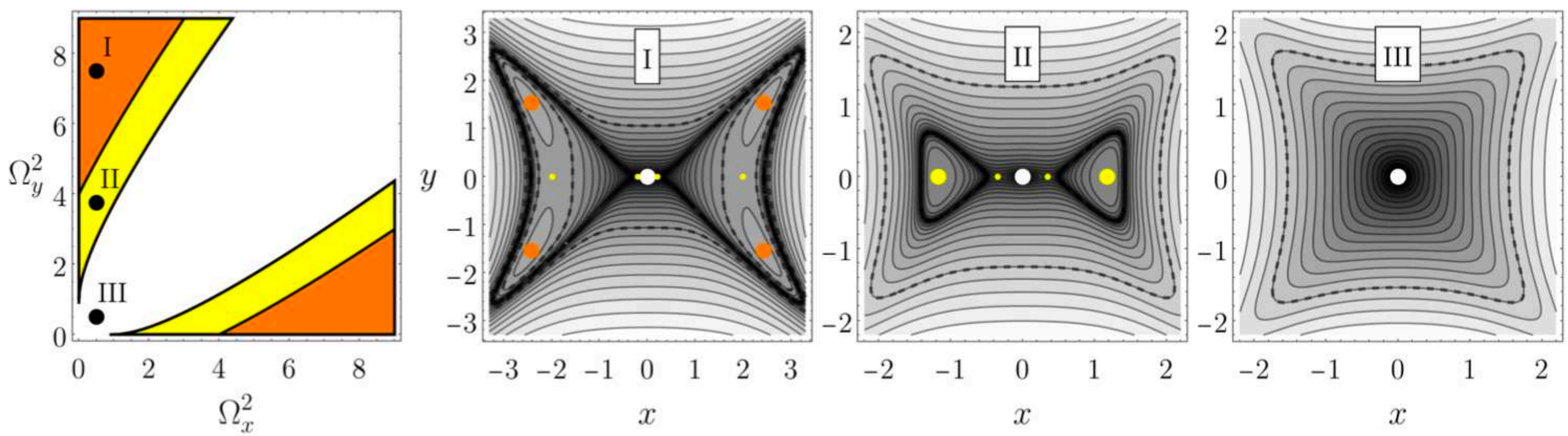
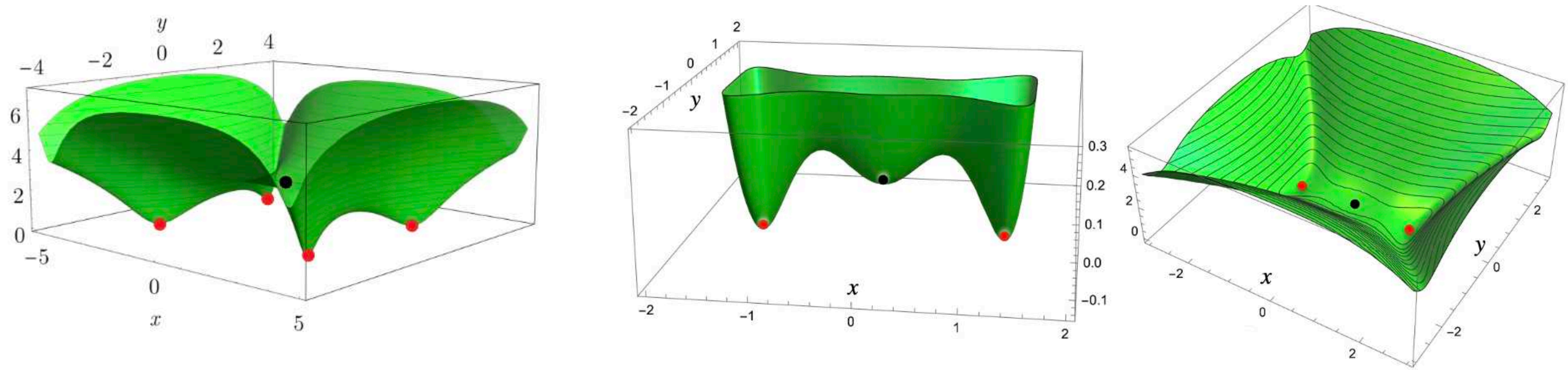


"Potential Energy" $\mathscr{U}_{LV}^{(4)}(x, y)$ and Vacua



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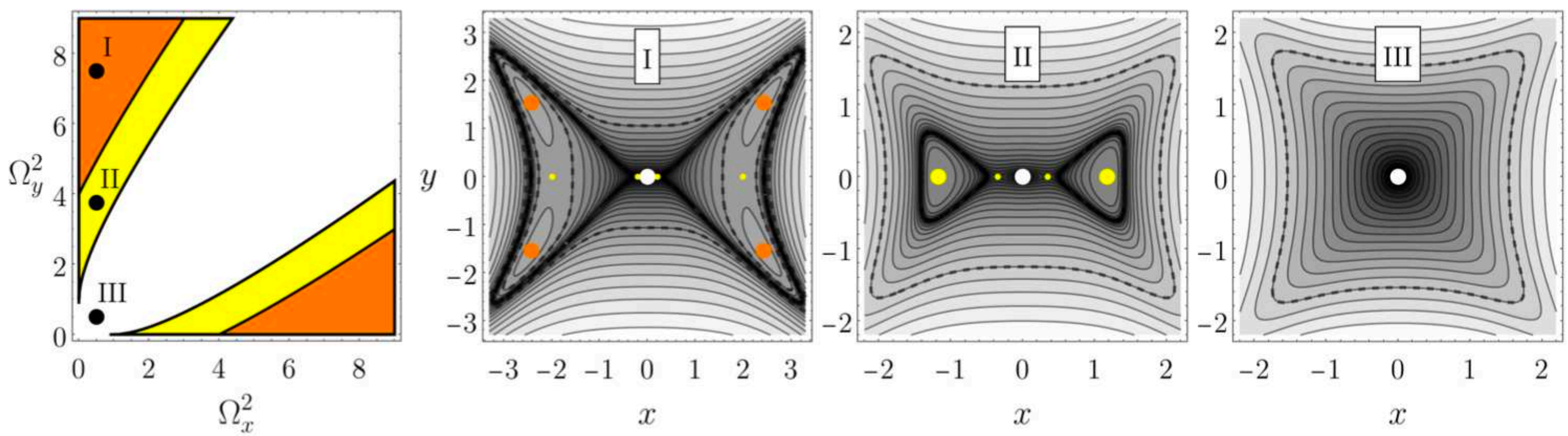
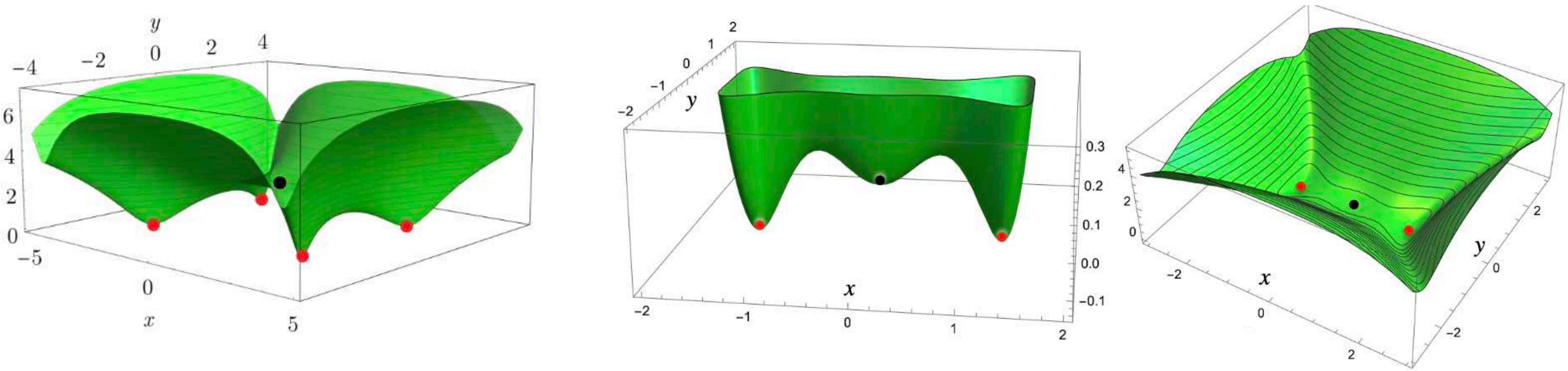


Minima of $\mathscr{U}$
are
saddle points of V

"Potential Energy" $\mathscr{U}_{LV}^{(4)}(x, y)$ and Vacua



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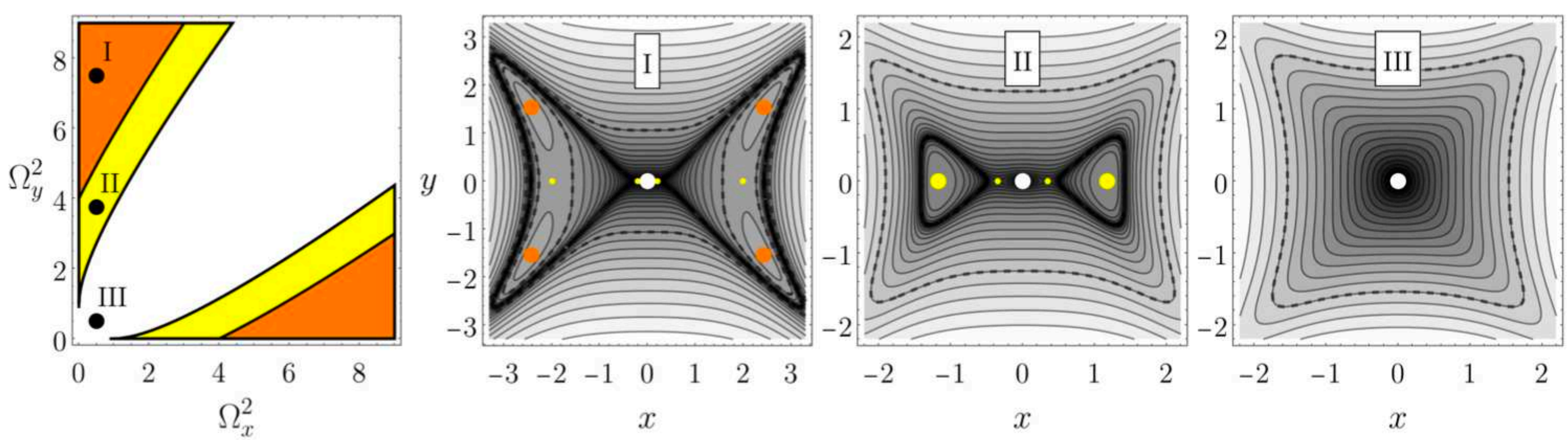

Minima of $\mathscr{U}$
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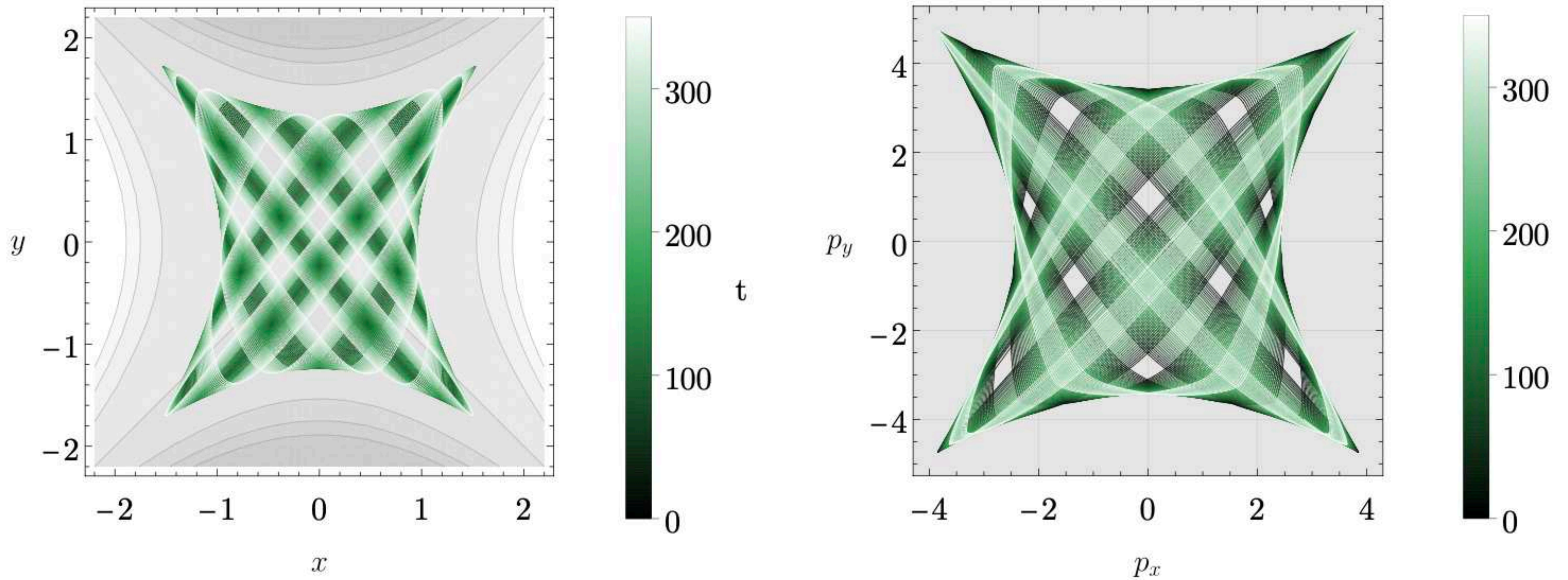
Interaction with ghost *creates* new Lyapunov stable vacua!

$$\omega_x^2 = 1\,, \omega_y^2 = 1\,, \mathscr{C}_4 = 1\,, c = 1$$



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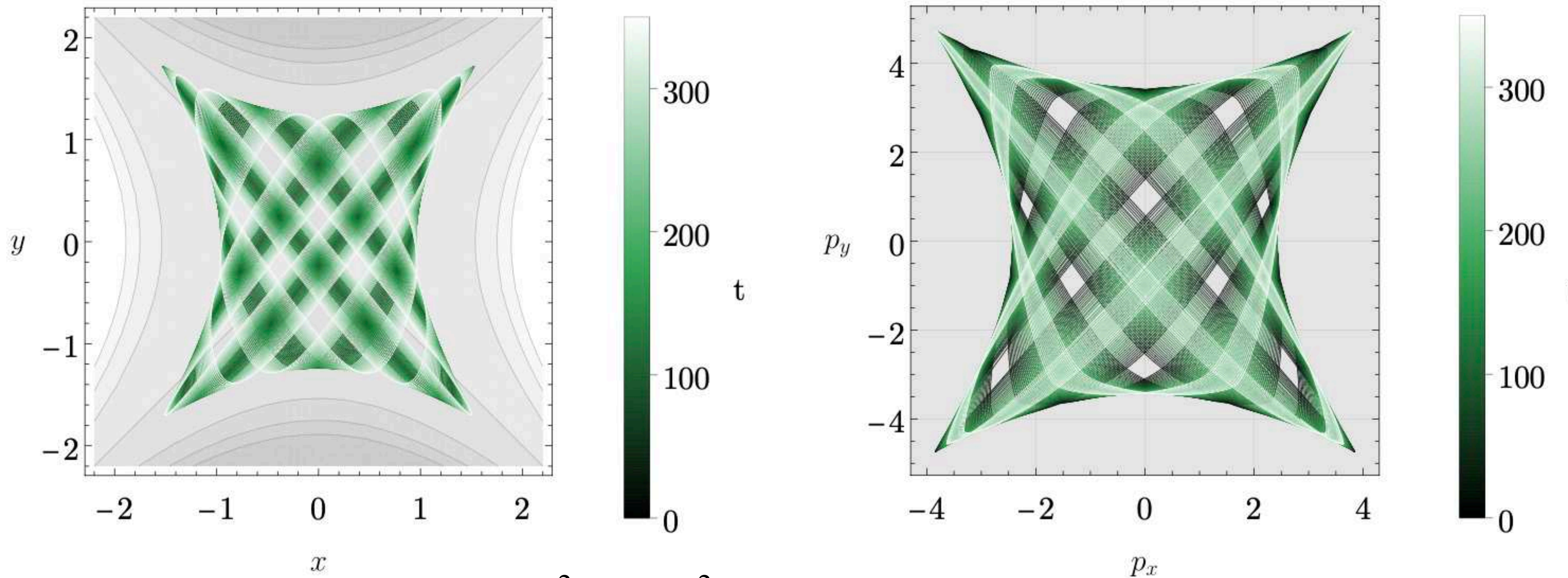
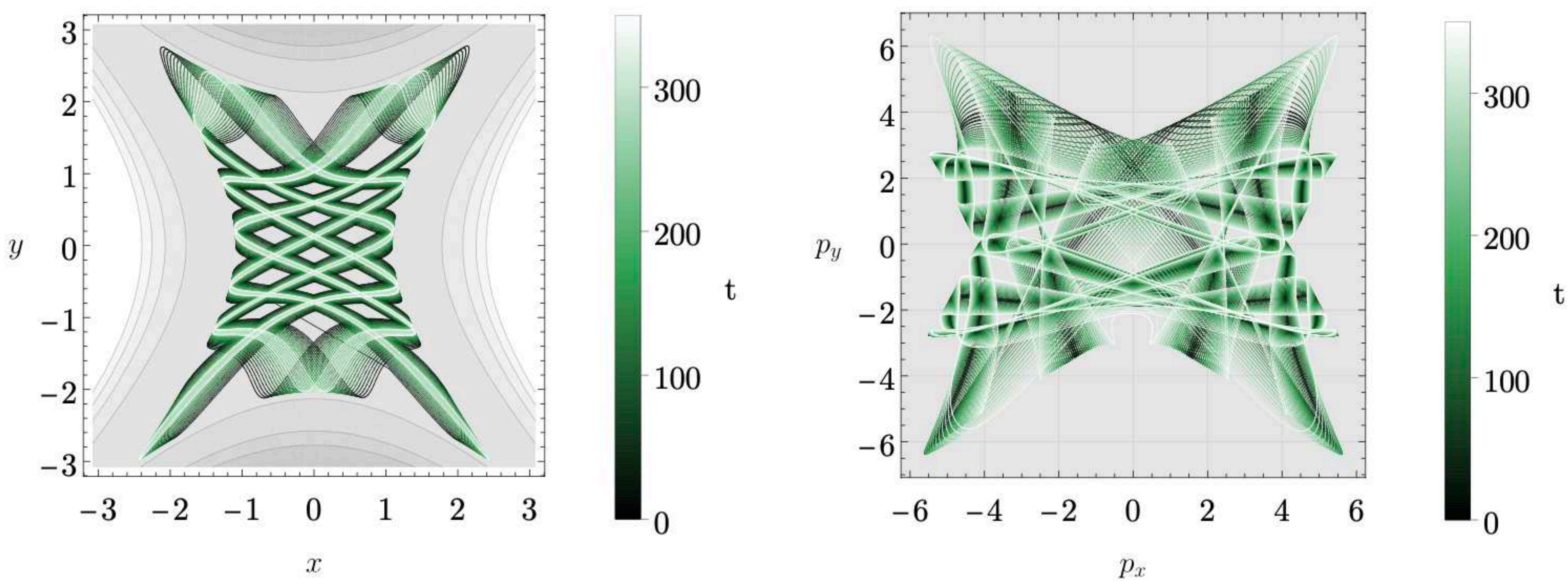


$$\omega_x^2 = 5\,, \omega_y^2 = -5\,, \mathscr{C}_4 = 1\,, c = 1$$



Kolmogorov–Arnold–Moser (KAM) theorem

↓

Small structural changes
do not jeopardise
the stability and finiteness
of motion

Why have not we seen
such systems in nature yet?


非常感谢您的关注！
Thanks a lot for attention!