Worldline Path Integrals for the Graviton and 1-loop Divergences in Quantum Gravity

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Outline

- Study 1-loop UV divergences in Quantum Gravity:
  systematize old results
  and get new ones with new methods
- Older methods: Schwinger-DeWitt heat kernel approach and worldline path integrals
- Graviton in first quantization: the $N = 4$ spinning particle

2307.09353 with Francesco Comberiati, Filippo Fecit, Fabio Ori (JHEP)

2305.06650 with Mattia Damia Paciarini (CQG)

2206.13287 with Roberto Bonezzi, Marco Melis (EPJC)

1909.05750 with Roberto Bonezzi, Olindo Corradini, Emanuele Latini (JHEP)

Quantum gravity

- Study 1-loop divergences of Einstein-Hilbert action in D euclidean dimensions

$$S[g_{\mu\nu}] = -\frac{1}{\kappa^2}\int d^D x\sqrt{g}\left[R(g) - 2\Lambda\right]$$

- Aim is to systematize old results and get new ones with novel methods
- $\kappa^2 = 16\pi G_N \sim M_{Pl}^{2-D}$ is the coupling constant, Λ cosmological constant
- Classical equations of motion

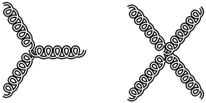
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

imply a constant scalar curvature: $R = \frac{2D}{D-2}\Lambda$
and Ricci tensor is proportional to metric tensor: $R_{\mu\nu} = \lambda g_{\mu\nu}$

• Spacetimes with such metrics are called Einstein spaces.

- Study divergences of Einstein-Hilbert action by viewing it as an interacting QFT, treated perturbatively in terms of Feynman diagrams built from propagators

and vertices



$\cdots$

- The coupling constant $k \sim M_{Pl}^{-\frac{D}{2}+1}$ is of negative mass dimension and makes it into a nonrenormalizable QFT in $D \geq 4$
- How to characterize its UV divergences? Study divergencies at 1-loop

Quantum gravity

- Use background field method: rename $g_{\mu\nu} \to G_{\mu\nu}$

$$\boxed{S[G_{\mu\nu}] = -\frac{1}{\kappa^2}\int d^D x \sqrt{G}\left[R(G) - 2\Lambda\right]}$$

and split $G_{\mu\nu}(x) = \underbrace{g_{\mu\nu}(x)}_{\text{background}} + \underbrace{h_{\mu\nu}(x)}_{\text{quantum field}}$

- The 1-loop effective action $\Gamma[g]$ is given by the path integral

$$e^{-\Gamma[g]} = \int \frac{Dh}{Vol(Gauge)} e^{-S[g+h]}\Big|_{\text{quadratic in h}} = \underbrace{\mathrm{Det}^{-\frac{1}{2}} F}_{\text{graviton}} \; \underbrace{\mathrm{Det}\,\mathfrak{F}}_{\text{ghosts}} = e^{-}$$

- Thus, 1-loop effective action $\Gamma[g]$ is given by

$$\Gamma[g] =$$  $$= -\ln\left(\mathrm{Det}^{-\frac{1}{2}} F \; \mathrm{Det}\, \mathfrak{F}\right) = \frac{1}{2}\mathrm{Tr}\ln F - \mathrm{Tr}\ln\mathfrak{F}$$

- At this point, one may use the Schwinger-DeWitt heat kernel method by representing the logarithm in terms of a “proper time" T

$$\ln\frac{a}{b} = -\int_0^\infty \frac{dT}{T}(e^{-aT} - e^{-bT})$$

and extending the formula to operators. Dropping an infinite constant one finds the *effective action* in terms of *heat kernels*

$$\Gamma[g] =$$  $$= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\left(\mathrm{Tr}\left[\mathrm{e}^{-FT}\right] - 2\mathrm{Tr}\left[\mathrm{e}^{-\mathfrak{F}T}\right]\right)$$

• $\Gamma[g]$ is in general *gauge dependent*, but it is *gauge invariant* on-shell

Use background equations of motions ($g_{\mu\nu}$ metrics of Einstein spaces) to get gauge invariant results.

• To identify divergences, one calculates $\Gamma[g]$ using an expansion for small proper time T of the heat kernels

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\left(\mathrm{Tr}\left[e^{-FT}\right] - 2\mathrm{Tr}\left[e^{-\mathfrak{F}T}\right]\right)$$

$$\approx \int d^Dx\sqrt{g(x)}\left[-\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2T}}{(4\pi T)^{\frac{D}{2}}}\sum_{n=0}^{\infty} a_n(x)\,T^n\right]$$

the mass m^2 is an IR regulator.

• The small T expansion gives the 1-loop UV divergences (arising from the $T \to 0$ integration limit) in terms of the Seeley-DeWitt coefficients $a_n(x)$

$$D = 4 \quad \rightarrow \quad a_2, a_1, a_0$$

$$D = 6 \quad \rightarrow \quad a_3, a_2, a_1, a_0$$

$$\Gamma[g] \approx \int d^D x \sqrt{g(x)} \left[-\frac{1}{2} \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \sum_{n=0}^{\infty} a_n(x) T^n \right]$$

We see that in even D dimensions there are divergences in $T \to 0$ region.
Integrate term-by-term in the proper-time T to find the gamma function $\Gamma(x)$

$$\int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} T^n = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{1}{(m^2)^{n-\frac{D}{2}}} \Gamma\left(n - \frac{D}{2}\right)$$

Recognize diverging terms:

$$\begin{array}{lcl} D = 4 & \to & \text{divergences for } n = 0, 1, 2, \\ D = 6 & \to & \text{divergences for } n = 0, 1, 2, 3 \\ \ldots & & \ldots \\ D & \to & \text{divergences for } n = 0, 1, 2, 3, \cdots, \frac{D}{2}. \end{array}$$

Evaluated on-shell, the metric must satisy $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$,
i.e. $\Lambda = \frac{D-2}{2D} R$ and $R_{\mu\nu} = \frac{1}{D} g_{\mu\nu} R$, coefficients become **gauge invariant**

$$a_0 = \frac{D(D-3)}{2} \qquad \text{(number of degrees of freedom of the graviton)}$$

$$a_1 = \frac{D^2 - 3D - 36}{12} R$$

$$a_2 = \frac{5D^3 - 17D^2 - 354D - 720}{720D} R^2 + \frac{D^2 - 33D + 540}{360} R^2_{\mu\nu\rho\sigma}$$

$$\begin{aligned} a_3 = &\frac{35D^4 - 147D^3 - 3670D^2 - 13560D - 30240}{90720D^2} R^3 \\ &+ \frac{7D^3 - 230D^2 + 3357D + 12600}{15120D} R\, R^2_{\mu\nu\rho\sigma} \\ &+ \frac{17D^2 - 555D - 15120}{90720} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} \\ &+ \frac{D^2 - 39D - 1080}{3240} R_{\alpha\mu\nu\beta} R^{\mu\rho\sigma\nu} R_{\rho}{}^{\alpha\beta}{}_{\sigma} \end{aligned}$$

New result!

Values at $D = 4$

$$a_0 \xrightarrow{D=4} 2$$

$$a_1 \xrightarrow{D=4} -\frac{8}{3} R$$

$$a_2 \xrightarrow{D=4} -\frac{29}{40} R^2 + \frac{53}{45} R^2_{\mu\nu\rho\sigma} \qquad \text{(log div)}$$

- 't Hooft-Veltmann '74 ($\Lambda = 0$ i.e. $R = 0$): $a_2 = 0$ up to total derivatives
  $\rightarrow$ QG renomalizable at one-loop
- Gibbons-Hawking-Perry '78: coefficient of topological $R^2_{\mu\nu\rho\sigma}$ term
- Christensen-Duff '80 ($\Lambda \neq 0$ i.e. $R \neq 0$): $a_2 \neq 0$
  $\rightarrow$ QG with cosm. const. is non-renomalizable
- UV divergences at 2-loop (Goroff-Sagnotti '86, van de Ven '92) or at 1-loop with generic matter fields
- search for improved UV theories: supergravities, superstrings, and other more recent proposals

Values at $D = 6$

$$a_0 \xrightarrow{D=6} 9$$

$$a_1 \xrightarrow{D=6} -\frac{3}{2}R$$

$$a_2 \xrightarrow{D=6} -\frac{11}{20}R^2 + \frac{21}{20}R^2_{\mu\nu\rho\sigma}$$

$$a_3 \xrightarrow{D=6} -\frac{799}{11340}R^3 + \frac{481}{1680}RR^2_{\mu\nu\rho\sigma} + \frac{991}{5040}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} - \frac{71}{180}R_{\alpha\mu\nu\beta}R^{\mu\rho\sigma\nu}R_{\rho}{}^{\alpha\beta}{}_{\sigma}$$

Comparison with the literature: results only known for $\Lambda = 0$
Thus, set $\Lambda = 0$ (i.e. $R = 0$) and use Gauss-Bonnet theorem

$$a_3\Big|_{\substack{D=6\\ \Lambda=0}} = -\frac{9}{15120}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}$$

agreement with:
van Nieuwenhuizen ’77, Chritchley ’78;
Gibbons-Ichinose 2000, Dunbar-Tuner 2003

Gauge invariant coefficients

$$a_0 = \frac{D(D-3)}{2}$$

$$a_1 = \frac{D^2 - 3D - 36}{12} R$$

$$a_2 = \frac{5D^3 - 17D^2 - 354D - 720}{720D} R^2 + \frac{D^2 - 33D + 540}{360} R^2_{\mu\nu\rho\sigma}$$

$$a_3 = \frac{35D^4 - 147D^3 - 3670D^2 - 13560D - 30240}{90720D^2} R^3$$

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$$+ \frac{17D^2 - 555D - 15120}{90720} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}$$

$$+ \frac{D^2 - 39D - 1080}{3240} R_{\alpha\mu\nu\beta} R^{\mu\rho\sigma\nu} R_{\rho}{}^{\alpha\beta}{}_{\sigma}$$

• For $D \geq 8$, previous coefficients provide only a subset of the divergences of quantum gravity

• Coefficients are *gauge invariant* and *characterize* quantum gravity at 1-loop

• Benchmark for alternative approaches to (perturbative) quantum gravity

We have used three methods for their calculation:

1. Standard Schwinger-DeWitt heat kernel method
2. Worldline path integrals for computing the heat kernel
3. First-quantized graviton: the N=4 spinning particle

Methods 1 and 2: Heat kernel and worldlines

In the Schwinger-DeWitt heat kernel method we had

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\left(\mathrm{Tr}\,[\mathrm{e}^{-FT}] - 2\mathrm{Tr}\,[\mathrm{e}^{-\mathfrak{F}T}]\right)$$

where the differential operators

$$\begin{aligned} F_{\mu\nu}{}^{\sigma\tau} &= -\delta_\mu^{(\sigma}\delta_\nu^{\tau)}(\nabla^2 + 2\Lambda - R) - 2\delta_{(\mu}^{(\sigma}R_{\nu)}^{\tau)} - 2R_\mu{}^{(\sigma}{}_\nu{}^{\tau)} \\ &\quad - \frac{1}{D-2}g_{\mu\nu}g^{\sigma\tau}R + \frac{2}{D-2}g_{\mu\nu}R^{\sigma\tau} + g^{\sigma\tau}R_{\mu\nu} \\ \mathfrak{F}^\mu{}_\nu &= -\delta_\nu^\mu\nabla^2 - R_\nu^\mu \end{aligned}$$

are interpreted as fictitious quantum mechanical Hamiltonians.

Expansion of the heat kernel trace for small proper time T has the form

$$\mathrm{Tr}[e^{-HT}] = \int d^D x \sqrt{g(x)} \sum_i \langle x, i | e^{-TH} | x, i \rangle$$
$$= \int d^D x \sqrt{g(x)} \left[\frac{1}{(4\pi T)^{\frac{D}{2}}} \sum_{n=0}^{\infty} a_n(x|H) T^n \right]$$

where the heat kernel coefficients $a_n(x|H)$ are sometimes called Seeley-DeWitt coefficients.

$a_n(x|H)$ can be computed by using recursive relations obtained from the heat equation (i.e. the Schrödinger eq. for imaginary time) satisfied by the kernel e^{-HT} (i.e. e^{-iHt} for $t = -iT$ that achieves the so-called Wick rotation). This is method 1.

As for method 2, use the equivalence of operatorial methods with path integrals to **compute** the heat kernels for the operators $H \equiv (F, \mathfrak{F})$

$$\langle x_f | e^{-TH} | x_i \rangle = \int_{x(0)=x_i}^{x(T)=x_f} Dx(\tau)\, e^{-S[x(\tau)]}$$

$$\mathrm{Tr}\left[e^{-TH}\right] = \int d^D x\, \langle x | e^{-TH} | x \rangle = \int_{PBC} Dx(\tau)\, e^{-S[x(\tau)]}$$

$S[x(\tau)]$ is the action corresponding to the quantum Hamiltonian H

$PBC \equiv$ Periodic Boundary Conditions

$\rightarrow$ Need to identify the actions S related to the operators H

Worldline actions for the Hamiltonians H arising from QG where constructed in $D = 4$ by separating a traceless graviton from its trace

$$h_{\mu\nu} \equiv \bar{h}_{\mu\nu} + \frac{1}{D} g_{\mu\nu}\, h$$

(FB, Roberto Bonezzi in 1304.7135, JHEP 07 (2013) 016)

and extended to arbitrary D dimensional backgrounds with Einstein metrics

$$R_{\mu\nu} = \lambda g_{\mu\nu}$$

to find the operators

$$H_h = -\nabla^2 - \frac{2R}{D}$$

$$(H_{bc})_\mu{}^\nu = -\left(\nabla^2 + \frac{R}{D}\right)\delta_\mu^\nu$$

$$(H_{\bar{h}})_{\mu\nu}{}^{\rho\sigma} = -\nabla^2 \delta_\mu^\rho \delta_\nu^\sigma - 2R_\mu{}^\rho{}_\nu{}^\sigma$$

(FB, Mattia Damia Paciarini in 2305.06650, CQG 41 (2024) 11, 115002)

1. The scalar particle

The scalar particle (the trace of the graviton) is the simplest system.
It shows almost all of the technical details of the worldline approach.

- In **flat space**, a free particle (of mass $m = \frac{1}{2}$) with Hamiltonian $H = -\partial^2 = p^2$ has classical action $S[x(\tau)] = \int_0^T d\tau \, \frac{1}{4}\dot{x}^2$ and gives rise to

$$\mathrm{Tr}\left[e^{-TH}\right] = \int_{PBC} Dx(\tau)\, e^{-S[x(\tau)]} = \int d^D x \, \frac{1}{(4\pi T)^{\frac{D}{2}}}$$

- In **curved space**, the Hamiltonian $H = -\nabla^2 + V(x)$ with laplacian

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu$$

has quantum ordering

$$H = g^{-\frac{1}{4}}(\hat{x})\, \hat{p}_\mu \, \sqrt{g(\hat{x})}\, g^{\mu\nu}(\hat{x})\, \hat{p}_\nu \, g^{-\frac{1}{4}}(\hat{x}) + V(\hat{x})$$

with classical Hamiltonian $H_{cl} = g^{\mu\nu}(x)\, p_\mu p_\nu + V(x)$ and action

$$S[x(\tau)] = \int_0^T d\tau \left(\frac{1}{4} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + V(x) \right)$$

For the particle related to the graviton trace we need a specific potential

$$H_h = -\nabla^2 - \frac{2R}{D}$$

and in the path integral we must use

$$S_h[x] = \int_0^T d\tau \left(\frac{1}{4} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu - \frac{2}{D} R + V_{ct} \right)$$

so that

$$\boxed{\Gamma_h[g] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \mathrm{Tr} \left[e^{-TH_h} \right] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int_{PBC} \mathcal{D}x \; e^{-S_h[x]}}$$

The **explicit perturbative path integral computation** leads to

$$\Gamma_h[g] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int \frac{d^D x \sqrt{g}}{(4\pi T)^{\frac{D}{2}}} \Big[1 + TR \left(\frac{D+12}{6D} \right) + T^2 R^2 \left(\frac{5D^2 + 118D + 720}{360 D^2} \right) + T^2 R^2_{\mu\nu\rho\sigma} \left(\frac{1}{180} \right) + \mathcal{O}(T^3) \Big]$$

Technical details

- counterterms
- nontrivial measure and measure ghosts
- factorization of zero modes

• Counterterms: the action $S_h[x] = \int_0^1 d\tau \left(\frac{1}{4T} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + ...\right)$ is a nonlinear sigma model with derivative interactions.
As a (0+1) QFT, it is super-renormalizable and needs *finite counterterms* to match renormalization conditions (i.e. require that K is the quantum Hamiltonian associated with it).
Here are the countertems associated with some reg. schemes

Time slicing $$V_{TS} = -\frac{1}{4}R + \frac{1}{4} g^{\mu\nu}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}$$

Mode regularization $$V_{MR} = -\frac{1}{4}R - \frac{1}{12} g^{\mu\alpha} g^{\nu\beta} g_{\rho\gamma}\Gamma^\rho_{\mu\nu}\Gamma^\gamma_{\alpha\beta}$$

Worldline dimensional regularization $$V_{DR} = -\frac{1}{4}R$$

- **Measure ghosts**: Express the covariant measure in path integral as

$$\mathcal{D}x = \prod_\tau \sqrt{g(x(\tau))}\, d^D x(\tau) = Dx \int DaDbDc\, e^{-S_{gh}[x,a,b,c]}$$

$$S_{gh}[x,a,b,c] = \int_0^1 d\tau \frac{1}{4T} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu)$$

where $Dx = \prod_\tau d^D x(\tau)$ is the translational invariant measure, etc. Note that a^μ is bosonic while b^μ, c^μ are fermionic.

In the sigma model this amounts to the shift

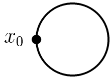
$$\dot{x}^\mu \dot{x}^\nu \to \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu$$

These "measure ghosts" give rise to divergences that compensate the divergences from the $\langle \dot{x}\dot{x} \rangle$ correlators, thus cancelling divergences on the worldline $\to$ in particular, the **counterterms are finite**

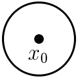
• Factorization of zero modes: With periodic boundary conditions on the path integral, can expand paths by

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$$

x_0^μ is the constant zero mode to be factored out and integrated at last (it remains as the spacetime integration of the effective lagrangian)
The two most commonly used methods are:
• Dirichlet boundary conditions method (DBC): $q^\mu(0) = q^\mu(1)$



• String-inspired method (SI): $\int_0^1 d\tau\, q^\mu(\tau) = 0$



2. The vector (ghost) particle

The ghost particle needs additional degrees of freedom on the worldline. To realize a *vector index* on the wave function on which H_{bc} acts, consider coordinates x^μ, p_μ and complex worldline fermions $\lambda^\mu, \bar{\lambda}_\mu$.

$$[x^\mu, p_\nu] = i\,\delta^\mu_\nu \,, \qquad \{\lambda^\mu, \bar{\lambda}_\nu\} = \delta^\mu_\nu$$

They act on the Hilbert space of antisymmetric tensor fields

$$|\Psi\rangle \sim \Psi(x,\lambda) = \sum_{n=0}^{D} \Psi_{\mu_1\cdots\mu_n}(x)\,\lambda^{\mu_1}\cdots\lambda^{\mu_n} \,.$$

Devise a projection to keep only $\Psi_\mu(x)$ in the physical space:
couple to a U(1) worldline gauge field $a(\tau)$ with specific Chern-Simons (CS) coupling s

Recall operator $(H_{bc})_\mu{}^\nu = -\left(\nabla^2 + \frac{R}{D}\right)\delta_\mu^\nu$
Action is

$$S_{bc}[x,\lambda,\bar{\lambda},a] = \int_0^1 d\tau \left[\frac{1}{4T} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \bar{\lambda}_\mu \left(D_\tau + ia\right)\lambda^\mu + V_{ct} + isa \right]$$

with CS coupling $s = 1 - \frac{D}{2}$. The path integral gives

$$\Gamma_{bc}[g] = \int_0^\infty \frac{dT}{T}\mathrm{Tr}\left[e^{-TH_{bc}}\right] = \int_0^\infty \frac{dT}{T}\int_{P/A} \frac{\mathcal{D}x D\lambda D\bar{\lambda} Da}{\mathrm{Vol(Gauge)}} e^{-S_{bc}[x,\lambda,\bar{\lambda},a]}$$

$$= \int_0^\infty \frac{dT}{T}\int \frac{d^Dx\sqrt{g}}{(4\pi T)^{\frac{D}{2}}}\Bigg[D + TR\left(\frac{D+6}{6}\right) + T^2R^2\left(\frac{5D^2+58D+180}{360D}\right) + T^2R^2_{\mu\nu\rho\sigma}\left(\frac{D-15}{180}\right) + \mathcal{O}(T^3)\Bigg]$$

3. The tensor particle (graviton)

Also the graviton needs additional degrees of freedom on the wl.
To realize symmetric indices on the wave function for $(H_{\bar{h}})_{\mu\nu}{}^{\rho\sigma}$, consider now complex worldline fermions which form traceless, symmetric, rank 2 tensors

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \qquad \{\psi^{ab}, \bar{\psi}_{cd}\} = \delta^a_c\delta^b_d + \delta^a_d\delta^b_c - \frac{2}{D}\delta^{ab}\delta_{cd}$$

They act on the Hilbert space composed of wave functions of the form

$$|\Psi\rangle \backsim \Psi(x,\psi) = \sum_{n=0}^{\frac{(D+2)(D-1)}{2}} \Psi_{(ab)_1\cdots(ab)_n}(x)\,\psi^{(ab)_1}\ldots\psi^{(ab)_n}$$

Again we need to project to occupation number 1 by coupling to a U(1) worldline gauge field $a(\tau)$

Thus, for the operator $(H_{\bar{h}})_{\mu\nu}{}^{\rho\sigma} = -\nabla^2 \delta^\rho_\mu \delta^\sigma_\nu - 2R_\mu{}^\rho{}_\nu{}^\sigma$ the worldline action is

$$S_{\bar{h}}[x,\psi,\bar{\psi},a] = \int_0^1 d\tau \left[\frac{1}{4T} g_{\mu\nu}\dot{x}^\nu \dot{x}^\mu + \frac{1}{2}\bar{\psi}_{ab}(D_t + ia)\psi^{ab} - \frac{1}{2}R_{abcd}\psi^{ac}\bar{\psi}^{bd} + \cdots \right]$$

(dots refer to V_{ct} and CS term) and the path integral gives

$$\Gamma_{\bar{h}}[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\mathrm{Tr}\left[e^{-TH_{\bar{h}}}\right] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int_{P/A}\frac{\mathcal{D}xD\psi D\bar{\psi}Da}{\mathrm{Vol(Gauge)}} e^{-S_{\bar{h}}[x,\psi,\bar{\psi},a]}$$

$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int \frac{d^Dx\sqrt{g}}{(4\pi T)^{\frac{D}{2}}}\left[\frac{(D+2)(D-1)}{2} + TR\left(\frac{D^3+D^2-14D-24}{12D}\right)\right.$$

$$\left. + T^2R^2\left(\frac{5D^4+3D^3-132D^2-236D-1440}{720D^2}\right) + T^2R^2_{\mu\nu\rho\sigma}\left(\frac{D^2-29D+478}{360}\right) + \mathcal{O}(T^3)\right]$$

Summing all 3 contributions give the total a_0, a_1, a_2 given earlier

As for the a_3 coefficient, we have used the $N = 4$ spinning particle, to be discussed next

Method 3: graviton and the N=4 spinning particle

- A more principled way of treating the graviton in first-quantization
- One-loop effective action computed by a worldline path integral

$$\Gamma[g_{\mu\nu}] = \int_{S^1} \frac{\mathcal{D}X^\mu \, \mathcal{D}G}{\mathrm{Vol(Gauge)}} \, \mathrm{e}^{-S[X^\mu, G; g_{\mu\nu}]} =$$ 

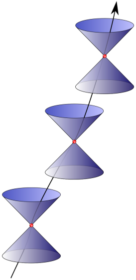
where $X^\mu = (x^\mu, \psi_i^\mu)$ and $G = (e, \chi_i, a_{ij})$ with $i = 1, ..., 4$
are the dynamical variables describing the graviton in first-quantization

- Preliminaries:

i) scalar bosonic relativistic particle ($N = 0$ model)
ii) spin 1/2 relativistic particle ($N = 1$ model)

$N = 0$ bosonic particle

Consider the particle's worldline $x^\mu(\tau)$



$$S[x,p,e] = \int d\tau \left(p_\mu \dot{x}^\mu - e \underbrace{\frac{1}{2}\left(p_\mu p^\mu + m^2\right)}_{H} \right)$$

$$\sim \int d\tau \left(\frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}_\mu - \frac{1}{2} e\, m^2 \right)$$

$$\sim -m \int d\tau \sqrt{-\dot{x}^2}$$

- reparametrization invariance: gauge symmetry generated by constraint H
- last form is the particle equivalent of the Nambu-Goto string action
- at the quantum level: constraint $H \;\rightarrow\;$ Klein-Gordon equation

$$H = 0 \qquad \rightarrow \qquad \hat{H}|\phi\rangle = 0 \;\sim\; (-\Box + m^2)\phi(x) = 0$$

$N = 1$ spinning particle

For the spin-1/2 particle need extra degrees of freedom to describe the spin.

The phase space action in the massless case is

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu - e \underbrace{\left(\frac{1}{2} p_\mu p^\mu\right)}_{H} - i\chi_i \underbrace{\left(p_\mu \psi^\mu\right)}_{Q} \right)$$

- Constraints H, Q generate a gauge symmetry: local $N = 1$ supersymmetry

$$\{Q, Q\} = -2iH$$

- At the quantum level: Q constraint $\rightarrow$ Dirac equation

$$Q = 0 \qquad \rightarrow \qquad \hat{Q}|\psi\rangle = 0 \;\sim\; \gamma^\mu \partial_\mu \psi(x) = 0$$

$N = 4$ spinning particle

N=4 spinning particle in flat space ($i = 1, ..., 4$)

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_{i\mu}\dot{\psi}_i^\mu - e\underbrace{\left(\frac{1}{2}p_\mu p^\mu\right)}_{H} - i\chi_i \underbrace{\left(p_\mu \psi_i^\mu\right)}_{Q_i} - \frac{1}{2}a_{ij}\underbrace{\left(i\psi_i^\mu \psi_{j\mu}\right)}_{J_{ij}} \right)$$

$$\{Q_i, Q_j\} = -2i\delta_{ij}H, \;\; \{J_{ij}, Q_k\} = \delta_{jk}Q_i - \delta_{ik}Q_j, \;\; \{J_{ij}, J_{kl}\} = \delta_{jk}J_{il} + \text{3 terms}$$

- Poincaré invariant $\rightarrow$ *relativistic particle*
- First class constraint algebra (N=4 susy algebra) $\rightarrow$ *gauge system*
- Graviton in $D = 4$, but not in other dimensions!
- Difficult to couple to backgrounds preserving first-class algebra
- BRST methods to get graviton for arbitrary D dimensional Einstein spaces by relaxing gauging of full R-Symmetry group SO(4) to parabolic subgroup (Bonezzi, Meyer, Sachs, JHEP 10 (2018) 025; arXiv: 1807.07989)

N=4 spinning particle and the graviton

Path integral for BRST model: upon gauge-fixing

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\,\mu(\theta,\phi)\int_P \mathcal{D}x\int_A D\bar{\psi}D\psi\, \mathrm{e}^{-S_g}$$

with gauge-fixed action ($i = 1, 2$ using complex fermions $\psi_i, \bar{\psi}^i$)

$$S_g = \int_0^1 d\tau\left[\frac{1}{4T}\,g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \bar{\psi}^{ia}(\delta_i^j D_\tau + i\hat{a}_i^j)\psi_{ja} - TR_{abcd}\,\bar{\psi}^a\cdot\psi^b\,\bar{\psi}^c\cdot\psi^d + TV_{0+ct}\right]$$

- Gauge field $\hat{a}_i^j = \begin{pmatrix} \theta & 0 \\ 0 & \phi \end{pmatrix}$ contains two moduli θ and ϕ
- Scalar potential $V_0 = -\frac{2}{D}R$ is an improvement term needed in the BRST charge. Also one needs a regularization $V_{ct} = \frac{1}{4}R$ in DR. Then total $V_{0+ct} = \frac{D-8}{4D}R$.
- The measure $\mu(\theta,\phi)$ coming from Faddeev–Popov determinants

$$\mu(\theta,\phi) = \frac{1}{2}\left(2\cos\frac{\theta}{2}\right)^{-2}\left(2\cos\frac{\phi}{2}\right)^{-2} 2i\,\sin\frac{\theta+\phi}{2}\,\left(2i\,\sin\frac{\theta-\phi}{2}\right)^2 \mathrm{e}^{-iq(\theta+\phi)}$$

includes the Chern-Simons coupling $q = \frac{3-D}{2}$

FB, Bonezzi, Corradini, Latini, JHEP 11 (2019) 124, arXiv:1909.05750;
FB, Bonezzi, Melis, Eur.Phys.J.C 82 (2022) 12, 1139, arXiv:2206.13287

- Then

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int d^Dx\,\frac{\sqrt{g}}{(4\pi T)^{\frac{D}{2}}}\left\langle\!\!\left\langle \mathrm{e}^{-S_{int}}\right\rangle\!\!\right\rangle$$

where $\langle\langle \ldots \rangle\rangle$ denotes the perturbative corrections of the path integral and subsequent modular integration.

- It is consistent only on Einstein spaces $\quad\leftrightarrow\quad Q^2_{BRST} = 0$
- The path integral gives an answer for Einstein spaces of the form

$$\left\langle\!\!\left\langle e^{-S_{int}}\right\rangle\!\!\right\rangle = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + O(T^4)$$

with the explicit coefficients computed up to this order and given earlier.

- Final result cross-checked with heat kernel methods

The heat-kernel at two worldline loops: an example of the calculations

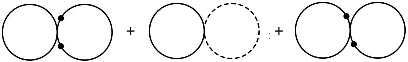
$$\langle x_0|e^{-TH}|x_0\rangle = \frac{1}{(4\pi T)^{\frac{D}{2}}}\langle \mathrm{e}^{-S_{int}}\rangle = \frac{1}{(4\pi T)^{\frac{D}{2}}}(1 - \langle S_{int}\rangle + \dots)\ , \qquad S[x] = \int_0^1 d\tau\, \frac{1}{4T} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$$

Set $x(\tau) = x_0 + q(\tau)$. To get order T^2 expand in Riemann normal coordinates and identify needed interactions

$$g_{\mu\nu}(x_0+q) = g_{\mu\nu}(x_0) + \frac{1}{3}R_{\alpha\mu\nu\beta}(x_0)q^\alpha q^\beta + \cdots \quad \Rightarrow \quad S_{int} = \frac{1}{12T}R_{\alpha\mu\nu\beta}(x_0)\int_0^1 d\tau\, q^\alpha q^\beta \dot{q}^\mu \dot{q}^\nu$$

Using the Wick contractions for the quantum field q with propagator $\langle q(\tau)q(\sigma)\rangle = -2T\Delta(\tau,\sigma)$ get

$$-\langle S_{int}\rangle = \frac{TR(x_0)}{3}\underbrace{\int_0^1 d\tau\,[\Delta({}^\bullet\Delta^\bullet + \Delta_{gh}) - {}^\bullet\Delta^2]|_\tau}_{-\frac{1}{4}} = -\frac{TR(x_0)}{12}$$

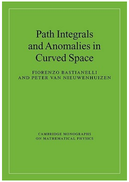


Thus

$$\langle x_0|e^{-\beta\hat{H}}|x_0\rangle = \frac{1}{(4\pi T)^{\frac{D}{2}}}\left[1 - \frac{TR(x_0)}{12} + O(T^2)\right]$$

Detailed discussions on path integral technicalities can be found in the book

F.B. and Peter van Nieuwenhuizen,
“Path Integrals and Anomalies in Curved Space" (CUP, 2006)



See also a forthcoming book

F.B. and Christian Schubert,
“Worldline Path Integrals and Quantum Field Theory" (CUP, expected 2025)

Conclusions

- Systematized old results in QG and computed the gauge invariant coefficients up to a_3 in arbitrary dimensions: characterization of QG at 1-loop
- Studied graviton in first quantization with the $N = 4$ particle
- Other applications of the $N = 4$ spinning particle?
- Other interesting BRST particle models?


That's all Folks!