

The Power of UV-Finite and Six-Derivative Quantum Gravitational Theories

Lesław Rachwał

Department of Physics, Institute for Exact Sciences (ICE)
Federal University of Juiz de Fora (UFJF)
(Juiz de Fora, MG, Brazil)



e-mail: grzerach@gmail.com

Quantum Gravity & Cosmology 2024

ShanghaiTech University, Shanghai, China
1-st – 5-th July 2024

This talk is based on collaboration with

prof. **Leonardo Modesto**

(Department of Physics, Southern University of Science and Technology (SUSTech), Shenzhen, China)

prof. **Aleksandr Pinzul**

(Institute of Physics, University of Brasília, Brasília, DF, Brazil)

prof. **Ilya Shapiro**

(Department of Physics - Institute for Exact Sciences, Federal University of Juiz de Fora, MG, Brazil)



and based on the paper **PRD 104, 085018 (2021)**

(ArXiv:hep-th/2104.13980 and ArXiv:hep-th/2204.09858)

Motivation for HD Gravity

Motivation:

Let's first quantize matter, put it on curved spacetime background, only later quantize gravitation (Utiyama, De Witt)

Observation:

1-loop off-shell divergences of standard matter theory (with two derivatives) are proportional to R^2 and C^2 on a curved spacetime background. Counterterms needed to be added to the divergent matter effective action are of these types R^2 and C^2 (in $d = 4$) even if the gravitational theory was Einstein-Hilbert Quantum Gravity with R in the action

Conclusion:

These counterterms contain higher derivatives of the background metric. Higher derivatives are inevitable!

Four-derivative theory (Stelle '77)

$$S_{\text{QG}} = \int d^4x \sqrt{|g|} (\omega_\kappa R + \theta_R R^2 + \theta_C C^2)$$

General higher-derivative theory (Asorey, Lopez, Shapiro '96)

$$S_{\text{QG}} = \int d^4x \sqrt{|g|} \left(\omega_\Lambda + \omega_\kappa R + \sum_{n=0}^N \omega_{R,n} R \square^n R + \sum_{n=0}^N \omega_{C,n} C \square^n C + O(\mathcal{R}^3) \right)$$

6-derivative theories

Here we consider the case $N = 1$.

We quantize the theory and study RG flow at one-loop level

We generalize Stelle's gravity and quantum results from it

Infinitely higher-derivative theory (Kuzmin, Tomboulis, Krasnikov, ...)

$$S_{\text{grav}} = \int d^4x \sqrt{|g|} (\lambda + \kappa_4^{-2} R + RF_R(\square)R + CF_C(\square)C)$$

Advantages:

- The most general theory describing gravitons' propagation
- Weakly non-local due to non-polynomial functions $F_R(\square)$ and $F_C(\square)$
- Unitary (optical theorem is satisfied, and not only)
- Propagator of gravitational modes is highly improved in UV
- In the spectrum only physical massless transverse graviton (spin 2) around flat or MSS background
- Asymptotically Free in UV, like Yang-Mills theory
- Good Quantum Loop Behaviour: renormalizable \rightarrow super-renormalizable \rightarrow UV-Finite

Super-renormalizable Quantum Gravity

Propagator of all quantum modes in UV regime (monomial asymptotics $F_i(\square) \rightarrow \square^\gamma$) **Modesto**

$$\Pi \sim k^{-(4+2\gamma)}$$

Superficial degree of divergence Δ of L -loop graph G

$$\Delta = 4L + V[\text{vertex}] - I[\text{propagator}]$$

Graviton $h_{\mu\nu}$ and FP ghost fields C_μ are dimensionless \Rightarrow the same maximal number of derivatives in vertices as in propagators in UV

$$[\text{vertex}] = -[\text{propagator}] = k^{4+2\gamma}$$

Bound on Δ

$$\Delta \leq 4 - 2\gamma(L - 1)$$

1-loop Super-renormalizability

For $\gamma \geq 3$ only 1-loop divergences survive

Example of an UV asymptotically polynomial form-factor (**Tomboulis**)

$$F = \frac{e^{H(-\square_\Lambda)} - 1}{\square} \quad \text{with} \quad H(z) = \frac{1}{2} [\Gamma(0, p(z)^2) + \gamma_E + \log(p(z)^2)],$$
$$z = -\square_\Lambda \equiv -\Lambda^{-2}\square, \quad \text{where } p(z) \text{ is UV polynomial}$$

Three (four) divergent contributions in the effective action:

$$\Gamma_{\text{div}} = \int d^4x \sqrt{|g|} (\beta_R R^2 + \beta_C C^2 + \beta_E E) + (\beta_{\square R} \square R)$$

Gravitational tensors and operators:

$$\text{Weyl square: } C^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3}R^2$$

$$\text{Gauss-Bonnet: } E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

$$\text{But } \delta \left(\int d^4x \sqrt{|g|} E \right) = 0$$

$$\text{GR-covariant d'Alembertian operator: } \square = g^{\mu\nu} \nabla_\mu \nabla_\nu$$

Ways for UV-Finiteness

To get full control over UV-behaviour of the effective action Γ_{eff} of the theory beta functions should vanish $\beta_i = 0$

But this should not be the effect of UV-regularization

Kuzmin '89:

- 1 relations between form-factors: $4F_R + 2F_{\text{Ric}} + 4F_{\text{Riem}} = 0$
and $F_{\text{Ric}} = \beta F_R$ with $\beta \approx -2.392$
- 2 asymptotically polynomial behaviour of the theory in UV regime
- 3 the particular irrational value of the exponent: $\gamma \approx 37.22$

FRG community:

In Wilsonian approach the effective action of a fundamental QFT should meet a non-trivial Fixed Point (FP) in the UV regime

UV-Finiteness (Main achievement)

Addition of killers operators

$$\Gamma_{\text{kill}} = s_1 R^2 \square^{\gamma-2} R^2 + s_2 C^2 \square^{\gamma-2} C^2 + s_3 F^2 \square^{\gamma-2} F^2 + (s_4 (\square R) \square^{\gamma-2} R^2),$$

where square of the trace-free Ricci tensor is $F^2 = R_{\mu\nu} R^{\mu\nu} - \frac{1}{4} R^2$

For UV-divergent action:

$$\Gamma_{\text{div}} = \int d^4x \sqrt{|g|} (\beta_R R^2 + \beta_C C^2 + \beta_F F^2) + (\beta_{\square R} \square R)$$

Linear structure of beta functions

$$\beta_i = v_i + a_{ij} s_j$$

The matrix a_{ij} is non-degenerate and diagonal

Condition for UV-Finiteness $\beta_i = 0$

solutions for non-running s_1, s_2, s_3 and s_4 always exist

Why 6-derivative Quantum Gravity?

Theoretical motivations

- quantum super-renormalizability
- possibility of UV-finiteness
- exact and unambiguous expressions for β -functions of running dimensionless coupling parameters of the theory
- gauge- and scheme-independence of UV-divergences
- possibility of Lee-Wick (LW) pair of complex conjugate poles of the propagator (to ameliorate the problem of unitarity in HD QG)
- *amazingly* simple and analytic final results for β -functions
- very good theoretical laboratory for study RG flows in QG

6-der QG

is better behaved on the quantum level than 4-der HD QG of Stelle!

Classical theory

Action:

$$S_{\text{QG}} = \int d^4x \sqrt{|g|} \mathcal{L},$$

Lagrangian (density):

$$\mathcal{L} = \omega_C C_{\mu\nu\rho\sigma} \square C^{\mu\nu\rho\sigma} + \omega_R R \square R + \theta_C C^2 + \theta_R R^2 + \theta_E E_4 + \omega_\kappa R + \omega_\Lambda$$

(some) GR invariant scalar terms:

$$C^2 = C_{\mu\nu\rho\sigma}^2 = R_{\mu\nu\rho\sigma}^2 - 2R_{\mu\nu}^2 + \frac{1}{3} R^2,$$

$$E_4 = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2.$$

Fundamental ratio of the theory:

$$\mathfrak{X} = \frac{\omega_C}{\omega_R}$$

Super-renormalizable Quantum Gravity

Propagator of all quantum modes in UV regime (monomial asymptotics \square^3) **Modesto**

$$\Pi \sim k^{-6}$$

Superficial degree of divergence Δ of L -loop graph G

$$\Delta = 4L + V[\text{vertex}] - I[\text{propagator}]$$

Graviton $h_{\mu\nu}$ and FP ghost fields C_μ are dimensionless \Rightarrow the same maximal number of derivatives in vertices as in propagators in UV

$$[\text{vertex}] = -[\text{propagator}] = k^6$$

Bound on Δ

$$\Delta \leq 4 - 2(L - 1); \quad \text{for } L \geq 4 \quad \Delta < 0$$

\Rightarrow no loop divergences for higher loops (quantum corrections are finite)

Consequences of power counting of UV divergences

Structure of divergences

- the only possible divergent structures are C^2 , R^2 , E_4 , R and Λ
- The divergences C^2 , R^2 , E_4 receive contributions only at one-loop level ($L = 1$)
- The R (Newton's gravitational constant) divergence receive contributions also at $L = 2$ level
- The Λ (cosmological constant) divergence receive contributions also at $L = 2, 3$ levels
- from 4-loop level the theory is finite
- terms θ_C , θ_R , θ_E , ω_κ and ω_Λ do not affect the counterterms C^2 , R^2 , E_4
- terms $O(\mathcal{R}^3)$ may affect above \implies possibility of complete UV-finiteness of the model

Here we concentrate on the most difficult to get counterterms
 C^2 , R^2 and E_4

Minimal model

Minimal working model

$$S_{\min} = \int d^4x \sqrt{|g|} \{ \omega_C C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \omega_R R^2 \}$$

The expected form of exact one-loop divergences

$$S_{\text{div}} = \int d^4x \sqrt{|g|} \{ c_C C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + c_R R^2 + c_E E_4 \}$$

Expected dependence

due to dimensional reasons

$$[c_C] = [c_R] = [c_E] = E^0, \quad [\mathfrak{X}] = \left[\frac{\omega_C}{\omega_R} \right] = E^0$$

$$c_C = c_C(\mathfrak{X}), \quad c_R = c_R(\mathfrak{X}), \quad c_E = c_E(\mathfrak{X})$$

Universality of UV-divergences in effective action Γ

Power counting for $L = 1$ $\Delta = 4$

- counterterm action S_{div} contains up to four derivatives on the metric
- classical minimal action S_{min} contains precisely six derivatives on the metric \implies classical EOM $\varepsilon^{\mu\nu}$ are with six derivatives

Parametrization independence theorem (Kallosh, Tyutin, Tarasov)

$$\Gamma(\alpha_i) - \Gamma(\alpha_i^0) = \int d^4x \sqrt{|g|} \varepsilon^{\mu\nu} f_{\mu\nu} \quad \text{with} \quad f_{\mu\nu} = f_{\mu\nu}(g_{\kappa\lambda}, \alpha_i, \alpha_i^0)$$

Independence of S_{div}

- of gauge choices
- of gauge-fixing choices
- of parametrization ambiguities for quantum field
- of scheme choice for renormalization

Method of computation

- covariant Barvinsky-Vilkovisky trace technology (generalized Schwinger-DeWitt method)
- quantum variable $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$
- based on the simple one-loop formula

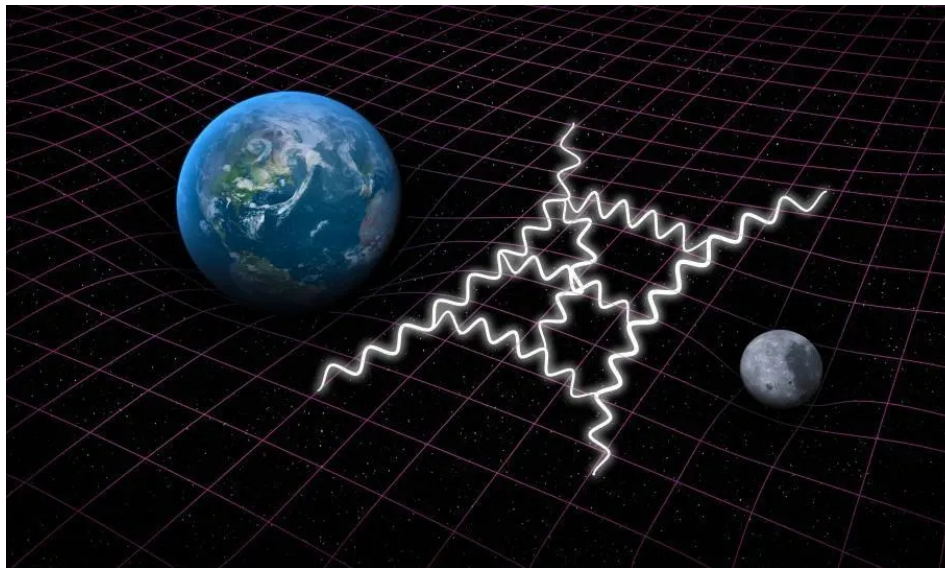
$$\Gamma^{(1)} = \frac{i}{2} \text{Tr} \ln \hat{H}, \quad \text{with} \quad \hat{H} = \frac{\delta^2 \mathcal{S}}{\delta \phi^2}$$

- minimal gauge fixing choice for gauge parameters
- simplified contributions from Faddeev-Popov and third ghosts quantum fields

$$\Gamma^{(1)} = \frac{i}{2} \text{Tr} \ln \hat{H} - i \text{Tr} \ln \hat{M} - \frac{i}{2} \text{Tr} \ln \hat{C}$$

- very difficult computation (needed to be done using Mathematica xTensor package for symbolic tensor algebra)
- various checks on it were successfully performed

Quantum computation in 6-derivative Gravitation



Results in 6-derivative gravitational theory in $d = 4$

$$\Gamma_{\text{div}}^{(1)C,R,E} = -\frac{1}{2\epsilon(4\pi)^2} \int d^4x \sqrt{|g|} \left\{ \left(\frac{2\mathfrak{X}}{9} + \frac{397}{40} \right) C^2 - \frac{7}{36} R^2 + \frac{1387}{180} E_4 \right\}$$

with $\epsilon = \frac{4-d}{2}$ as a parameter of DIMREG scheme
and the fundamental ratio $\mathfrak{X} = \frac{\omega_C}{\omega_R}$

Results in 4-derivative Stelle theory in $d = 4$

$$\Gamma_{\text{div}}^{(1)C,R,E} = -\frac{1}{2\epsilon(4\pi)^2} \int d^4x \sqrt{|g|} \left\{ -\frac{133}{20} C^2 + \left(-\frac{5}{2} \mathfrak{X}'^2 + \frac{5}{2} \mathfrak{X}' - \frac{5}{36} \right) R^2 + \frac{196}{45} E_4 \right\} \quad \text{with} \quad \mathfrak{X}' = \frac{\theta_R}{\theta_C}$$

impossibility to get full UV-finiteness at 1-loop ! (FT, Avramidi, Barvinsky,

System of β functions

$$\beta_C = \mu \frac{d\theta_C}{d\mu} = \frac{1}{(4\pi)^2} \left(\frac{2\omega_C}{9\omega_R} + \frac{397}{40} \right), \quad \text{exact}$$

$$\beta_R = \mu \frac{d\theta_R}{d\mu} = -\frac{1}{(4\pi)^2} \frac{7}{36}, \quad \text{exact}$$

$$\beta_E = \mu \frac{d\theta_E}{d\mu} = \frac{1}{(4\pi)^2} \frac{1387}{180}, \quad \text{exact}$$

$$\beta_\kappa = \mu \frac{d\omega_\kappa}{d\mu} = -\frac{1}{(4\pi)^2} \left[\frac{5\theta_C}{6\omega_C} + \frac{\theta_R}{2\omega_R} - \frac{5\theta_R}{\omega_C} \right],$$

$$\beta_\Lambda = \mu \frac{d\omega_\Lambda}{d\mu} = \frac{1}{(4\pi)^2} \left[\frac{5\omega_\kappa}{2\omega_C} - \frac{\omega_\kappa}{6\omega_R} - \frac{5}{2} \left(\frac{\theta_C}{\omega_C} \right)^2 - \frac{1}{2} \left(\frac{\theta_R}{\omega_R} \right)^2 \right].$$

$$\theta_C(t) = \theta_C(0) + \beta_C t = \theta_C(0) + \frac{1}{(4\pi)^2} \left(\frac{2\mathfrak{X}}{9} + \frac{397}{40} \right) t,$$

$$\theta_R(t) = \theta_R(0) + \beta_R t = \theta_R(0) - \frac{1}{(4\pi)^2} \frac{7}{36} t,$$

$$\theta_E(t) = \theta_E(0) + \beta_E t = \theta_E(0) + \frac{1}{(4\pi)^2} \frac{1387}{180} t,$$

$$\omega_\kappa(t) = \omega_\kappa(0) + a_\kappa t + b_\kappa t^2,$$

$$\omega_\Lambda(t) = \omega_\Lambda(0) + a_\Lambda t + b_\Lambda t^2 + c_\Lambda t^3.$$

Observation

For $t \rightarrow +\infty$ couplings θ_R and θ_E tend to $-\infty$ and $+\infty$ respectively (we have asymptotic freedom in them if $\theta_R(0) < 0$ and $\theta_E(0) > 0$); the coupling θ_C is also AF in UV, if not the special value of ratio \mathfrak{X} :

$$\mathfrak{X}_* = -\frac{3573}{80} = -44.6625.$$

For $\mathfrak{X} = \mathfrak{X}_*$ the coupling θ_C sits at the non-trivial FP (asymptotic safety)

Asymptotic Safety in Six-derivative Gravity

Condition for AS: $\beta_a = 0$

- in the sector of only C^2 counterterms
- the special value of the fundamental ratio
 $\mathfrak{X}_* = -\frac{3573}{80} = -44.6625$
- for $\mathfrak{X} = \mathfrak{X}_*$ the coupling θ_C sits at the non-trivial FP of RG (asymptotic safety)
- the value of θ_C there is arbitrary (1-dimensional line of FP's)

Conditions for AF in the C^2 sector

- if $\mathfrak{X} < \mathfrak{X}_*$, then $\theta_C(0) < 0$
- if $\mathfrak{X} = -\frac{3573}{80}$, then no RG flow in C^2 sector
- if $\mathfrak{X} > \mathfrak{X}_*$, then $\theta_C(0) > 0$

Can we have full UV-finiteness (\iff Asymptotic Safety) in the six-derivative QG model?

Conformal symmetry

UV-Finiteness equivalent to vanishing of conformal anomaly

$T \sim \beta$, where T is a trace of energy pseudo-tensor of the quantized gravitational field at one loop

$$T = \alpha_C C^2 + \alpha_E E + \alpha_{\square R} \square R = 0$$

$\beta_i = 0 \Rightarrow$ scale-invariance of Green functions \Rightarrow Conformal Invariance

Conformal symmetry in a consistent QFT of gravitational interactions

Conformal QG at UV fixed point of Renormalization Group

Conformal symmetry makes GR-like gravitational singularities conformal gauge-dependent, hence physically unobservable

Choice of the conformal factor $\Omega^2(x)$ gives: $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$

Why CFT?

Enhancement of symmetries compared to standard QFT

Solution to unitarity problem and unitarity bound

Model QFT with very good behaviour at any energy scale

Basis for doing conformal perturbation theory

Famous examples:

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory

$\mathcal{N} = 8$ conformal supergravity [Fradkin, Tseytlin](#)

Relations:

- to AdS/CFT correspondence (open–closed string theory duality)
- to string theory and theory of critical phenomena (CMT)

Why CFT of QG?

Very strong constraint on effective action

if there are no mass scales in the theory

Constraints on anomalous dimensions of operators

Only finite renormalization of couplings

No need for renormalization scale μ (arbitrary)

Idempotency of quantization procedure

Conformally symmetric phase

Basis for study various deformations

Cubic killers

Terms cubic

- non-minimal terms of the six-derivative gravitational action
- cubic in gravitational curvatures $\sim O(\mathcal{R}^3)$
- allowed by EFT of gravity (still with six derivatives)
- killer Lagrangian

$$\begin{aligned}\mathcal{L}_K = & \tilde{s}_1 R^3 + \tilde{s}_2 R R_{\mu\nu} R^{\mu\nu} + \tilde{s}_3 R_{\mu\nu} R^\mu{}_\rho R^{\nu\rho} + \tilde{s}_4 R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + \tilde{s}_5 R_{\mu\nu} R_{\rho\sigma} R^{\mu\rho\nu\sigma} + \tilde{s}_6 R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\kappa\lambda} R^{\rho\sigma\kappa\lambda}\end{aligned}$$

- general second order dependence $\beta_a \sim \tilde{s}_i \tilde{s}_j$

Example of full UV-finiteness

algebraic system of three equations of second order
killer coefficients \tilde{s}_i must be real

$$\omega_R = \omega_C, \quad \tilde{s}_1 = \tilde{s}_2 = 0, \quad \tilde{s}_3 = 5, \quad \frac{\tilde{s}_4}{\omega_C} \approx -0.847625, \quad \frac{\tilde{s}_5}{\omega_C} \approx 2.1177, \quad \frac{\tilde{s}_6}{\omega_C} \approx -9.83078$$

Killer Lagrangian

$$\begin{aligned}\mathcal{L}_K = & s_1 R^3 + s_2 R F_{\mu\nu} F^{\mu\nu} + s_3 F_{\mu\nu} F^\mu{}_\rho F^{\nu\rho} + s_4 R C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \\ & + s_5 F_{\mu\nu} F_{\rho\sigma} C^{\mu\rho\nu\sigma} + s_6 C_{\mu\nu\rho\sigma} C^{\mu\nu}{}_{\kappa\lambda} C^{\rho\sigma\kappa\lambda}\end{aligned}$$

Notation

$$\mathfrak{X} = \frac{\omega_C}{\omega_R}, \quad t_i = \frac{s_i}{\omega_C}$$

Dependencies:

$$\beta_R = \beta_R(t_1, t_2, t_4, \mathfrak{X})$$

$$\beta_F = \beta_F(t_2, t_3, t_5, \mathfrak{X})$$

$$\beta_C = \beta_C(t_4, t_5, t_6, \mathfrak{X})$$

Results for cubic killers

Algebraic system of beta functions

Condition for complete UV-finiteness: $\beta_R = \beta_F = \beta_C = 0$

Reality of coupling constants: $t_1, t_2, t_3, t_4, t_5, t_6, \mathfrak{X} \in \mathbb{R}$

Solved for t_1, t_3 and t_6 : $\tilde{\Delta}_1 \geq 0, \tilde{\Delta}_2 \geq 0, \tilde{\Delta}_3 \geq 0$

Intersection of three 3D regions given by algebraic inequalities,
intersection of three elliptical cylinders

Excluded range for \mathfrak{X}

$$\begin{aligned} & \text{Root} [x^5 - 297x^4 - 78432x^3 + 1415196x^2 + 7245315x + 1913625, 4] \leq \mathfrak{X} \leq \\ & \leq \text{Root} [55x^8 - 55835x^7 - 11660747x^6 + 26974221x^5 - 1543030335x^4 + \\ & + 1016620875x^3 - 4650868125x^2 - 478406250x - 3588046875, 2] \end{aligned}$$

or $20.9602 \lesssim \mathfrak{X} \lesssim 1192.62 \implies$ no UV-finiteness with real coefficients

Six-derivative Gravity

- super-renormalizability and options for UV-finiteness
- exact and universal beta functions for θ_C , θ_R and θ_E couplings
- gauge- and parametrization-independence of UV divergences
- exact RG flows and asymptotic freedom in UV

Further developments

- conditions for AF in UV and AS
- dominance of free propagation over interactions
- rescaling of the graviton field (like [Fradkin, Tseytlin](#))
- quantum stability of the Lee-Wick complex conjugate pairs
- addition of terms $O(\mathcal{R}^3)$ for UV-finiteness
- spectrum around flat Minkowski and around (A)dS spacetimes

- 1 “Effective action in quantum gravity”, I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, Bristol, UK: IOP (1992) 413 p
- 2 “Conformal Supergravity”, E.S. Fradkin, A.A. Tseytlin, Phys. Rept. **119** (1985) 233-362
- 3 “Renormalizable asymptotically free quantum theory of gravity”, E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. **B201** (1982) 469-491
- 4 “Some remarks on high derivative quantum gravity”, M. Asorey, J.L. López and I.L. Shapiro, Int. Journ. Mod. Phys. **A12** (1997) 5711
- 5 “Renormalization group in super-renormalizable quantum gravity”, L. Modesto, L. Rachwał and I.L. Shapiro, Eur. Phys. J. **C78** (2018) 555, hep-th/1704.03988
- 6 “Renormalization Group in Six-derivative Quantum Gravity”, L. Modesto, A. Pinzul, L. Rachwał and I.L. Shapiro, hep-th/2104.13980

谢谢

Thank you!

Dziękuję!

Obrigado!



谢谢

Thank you!