Axial Gravitational Perturbations of Slowly-Rotating Compact Objects

Jun Peng

Beijing University of Technology

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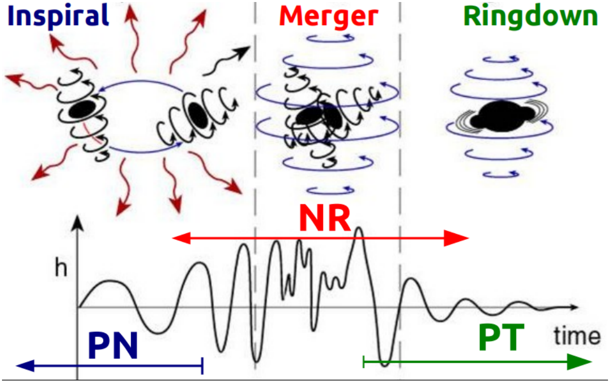
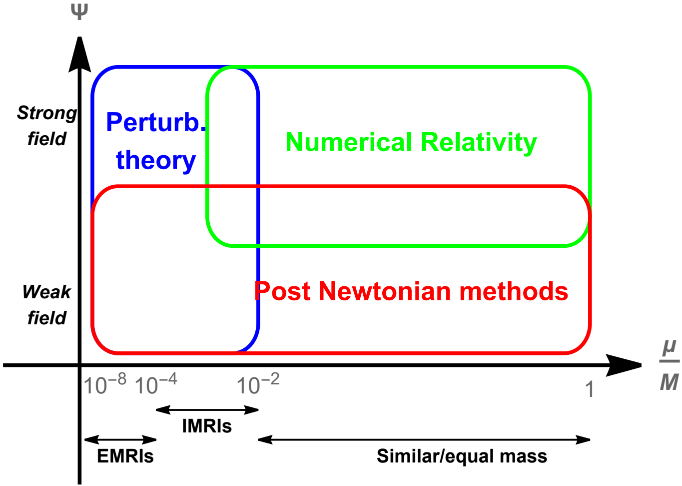
Why black hole dynamics is important?

- Stability
- No-hair theorem
- Hawking radiation
- Gravitational wave
- AdS/CFT
- $\cdots$


Inspiral
Merger
Ringdown
NR
h
PN
PT
time

Two body problem in GR



Index split

The perturbations of a spherically symmetric spacetime are expressed as

$$ds^2 = \underbrace{-h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-2}^2}_{\text{background}} + \underbrace{h_{\mu\nu}dx^\mu dx^\nu}_{\text{perturbations}} \tag{1}$$

Any spherically symmetric spacetime with dimension n can be decomposed as the product of a two-dimensional manifold $\mathbb{M}_2$ described by the coordinates (t, r), and the $(n-2)$-sphere $\mathbb{S}^{n-2}$, $\mathbb{M}_2 \times \mathbb{S}^{n-2}$. We split the index as

$$a = t, r \quad i = \theta, \phi, \cdots \tag{2}$$

With this decomposition, the components of a vector field V^μ are split as

$$V^\mu = (V^a, V^i) \tag{3}$$

The components of a rank-two symmetric tensor field $X_{\mu\nu}$ are split as

$$X_{\mu\nu} = \begin{pmatrix} X_{ab} & X_{ai} \\ X_{ia} & X_{ij} \end{pmatrix} \tag{4}$$

SVT decomposition of metric perturbations

The h_{ai} can be decomposed as

$$h_{ai} = D_i \Psi_a + \hat{h}_{ai}, \quad D^i \hat{h}_{ai} = 0 \tag{5}$$

where D is the covariant derivative with respect to the metric δ_{ij} on $\mathbb{S}$. The h_{ij} can be decomposed as

$$\begin{aligned} h_{ij} &= \Phi\gamma_{ij} + \left(D_i D_j - \frac{1}{n-2}\gamma_{ij}\Delta\right)\Psi + D_j\hat{\Psi}_j + D_j\hat{\Psi}_i + \hat{h}_{ij}, \\ D^i\hat{\Psi}_i &= 0, \quad D^i\hat{h}_{ij} = 0, \quad \gamma^{ij}\hat{h}_{ij} = 0 \end{aligned} \tag{6}$$

where $\Delta = \gamma^{ij} D_i D_j$ is the Laplacian for $\mathbb{S}^{n-2}$. Hatted quantities are divergence-free or transverse. Eventually we can classify all variables into three types according to their transformation law under the $SO(n-1)$ rotational symmetry of $\mathbb{S}^{n-2}$:

- scalar type perturbations $h_{ab}, \Psi_a, \Psi, \Phi$
- vector type perturbation $\hat{h}_{ai}, \hat{\Psi}_i$
- tensor type perturbation $\hat{h}_{ij}$

Harmonic expansion in $D = 4$

In four dimension $\hat{h}_{ij} = 0$, we can expand the perturbations using spherical harmonics

$$\begin{aligned} h_{ab} &= H_{ab} Y \\ h_{ai} &= \mathcal{H}_a Y_i + h_a S_i \\ h_{ij} &= r^2 K \gamma_{ij} Y + r^2 G Y_{ij} + 2 h_2 S_{ij} \end{aligned} \tag{7}$$

Y_i are the polar vector harmonics

$$Y_i \equiv D_i Y \tag{8}$$

S_i are the axial vector harmonics

$$S_i \equiv -\epsilon_i{}^j D_j Y \tag{9}$$

Y_{ij} are the polar rank-two tensor harmonics

$$Y_{ij} \equiv \left[D_i D_j - \frac{1}{2} \gamma_{ij} \Delta \right] Y = \left[D_i D_j + \frac{l(l+1)}{2} \gamma_{ij} \right] Y \tag{10}$$

S_{ij} are the axial rank-two tensor harmonics

$$S_{ij} \equiv \frac{1}{2} (D_i S_j + D_j S_i) \tag{11}$$

Perturbation variables

- polar/even/electric (scalar type) perturbation:

$$H_{00}, H_{01}, H_{11}, \mathcal{H}_1, \mathcal{H}_2, G, K$$

- axial/odd/magnetic (vector type) perturbation:

$$h_0, h_1, h_2$$

Gauge freedom

Note that the perturbed Einstein equation is invariant under the gauge transformations

$$x^{\mu} \to x^{\mu} + \xi^{\mu}, \quad \delta h_{\mu\nu} = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} \tag{12}$$

In order to extract the dynamics of the physical degrees of freedom from the perturbed equations, the gauge freedom must be eliminated through fixing gauge or constructing gauge invariant variables. Usually, it is convenient to fix a gauge to eliminated gauge freedom. Conventionally, we choose the gauge

$$\Psi_a = \Psi = \hat{\Psi}_i = 0 \tag{13}$$

which corresponds to the Regge-Wheeler gauge

$$\mathcal{H}_0 = \mathcal{H}_1 = G = h_2 = 0 \tag{14}$$

Polar vs Axial

$$h_{\mu\nu}(t,r,\theta,\phi) = h^{-}_{\mu\nu}(t,r,\theta,\phi) + h^{+}_{\mu\nu}(t,r,\theta,\phi) \tag{15}$$

Polar sector

$$h^{+}_{\mu\nu} = \begin{pmatrix} hH_0 & H_1 & 0 & 0 \\ * & H_2/f & 0 & 0 \\ * & * & r^2 K & 0 \\ * & * & * & r^2 \sin^2\theta K \end{pmatrix} Y \tag{16}$$

Axial sector

$$h^{-}_{\mu\nu} = \begin{pmatrix} 0 & 0 & h_0 S_\theta & h_0 S_\phi \\ * & 0 & h_1 S_\theta & h_1 S_\phi \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \tag{17}$$

Regge-Wheeler/Zerilli equations

Introducing proper master variables $\Psi^{\pm}$, we can obtain Schrodinger-like equations

$$\frac{d^2\Psi^{\pm}}{dr_*^2} + [\omega^2 - V^{\pm}]\Psi^{\pm} = 0 \tag{18}$$

and the potentials are given by [Regge and Wheeler, 1957; Zerilli, 1970]

$$\begin{aligned} V_- &= \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right] \\ V_+ &= \left(1 - \frac{2M}{r}\right)\left[\frac{2\lambda(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2}\right] \end{aligned} \tag{19}$$

where $\lambda = (l-1)(l+2)/2$.

Quasi-normal modes

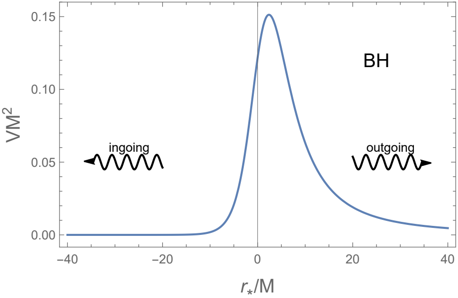


Figure: The effective potential for axial gravitational perturbation (l = 2) of a Schwarzschild BH. The effective potentials have a peak approximately at the photon sphere, $r = 3M$.

The eigenvalues of ω are complex with pure outgoing and ingoing boundary conditions.

$$\omega = \omega_{\rm Re} + i\omega_{\rm Im} \tag{20}$$

Isospectrality

The two potentials are generated by a *superpotential* [Chandrasekhar, 1975]

$$V^{\pm} = W^2(r) \pm \frac{dW(r)}{dr_*} + \beta \tag{21}$$

where $W(r)$ is a finite function and β is a constant. It can be shown that the solutions are related to each other by

$$\Psi_{\pm} \sim \left(\mp W + \frac{d}{dr_*} \right) \Psi_{\mp} \tag{22}$$

Therefore Ψ^+ and Ψ^- have the same eigenvalue ω.

Perturbations of Schwarzschild BH

The master wave equation for various perturbations takes the form

$$\frac{d^2\Psi^{(s)}}{dr_*^2} + (\omega^2 - V^{(s)})\Psi^{(s)} = 0 \tag{23}$$

where the potential is given by

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right) \tag{24}$$

where s is the spin of the perturbation being considered:

- $s = 0$ for scalar field
- $s = 1/2$ for spinor field
- $s = 1$ for electromagnetic field
- $s = 2$ for axial gravitational perturbation

Gravito-electromagnetic Perturbations of RN BH

We expand the electromagnetic potential as follows

$$\delta A_\mu = \begin{pmatrix} 0 \\ 0 \\ u_4 S_i \end{pmatrix} + \begin{pmatrix} u_1 Y \\ u_2 Y \\ u_3 Y_i \end{pmatrix} \tag{25}$$

- polar: u_1, u_2, u_3
- axial: u_4

Coupled equations for axial perturbations

$$\begin{aligned} \left(\frac{d^2}{dr_*^2} + \omega^2\right) H_2^- &= \frac{\Delta}{r^5}\left\{\left[l(l+1)r - 3M + 4\frac{Q^2}{r}\right] H_2^- - 3MH_2^- + 2Q\sqrt{(l-1)(l+2)}H_1^-\right\} \\ \left(\frac{d^2}{dr_*^2} + \omega^2\right) H_1^- &= \frac{\Delta}{r^5}\left\{\left[l(l+1)r - 3M + 4\frac{Q^2}{r}\right] H_1^- + 3MH_1^- + 2Q\sqrt{(l-1)(l+2)}H_2^-\right\} \end{aligned} \tag{26}$$

where $\Delta = r^2 - 2Mr + Q^2$, H_2^- corresponds to perturbations of the gravitational field and H_1^- to perturbations of the EM field.

Introduce

$$Z_1^- = +q_1 H_1^- + \sqrt{-q_1 q_2} H_2^- , \quad Z_2^- = -\sqrt{-q_1 q_2} H_1^- + q_1 H_2^- \tag{27}$$

where

$$q_1 = 3M + \sqrt{9M^2 + 4(l-1)(l+2)Q^2}, \quad q_2 = 3M - \sqrt{9M^2 + 4(l-1)(l+2)Q^2} \tag{28}$$

We can obtain decoupled Schrodinger-like equations [Moncrief, 1974]

$$\left(\frac{d^2}{dr_*^2} + \omega^2 \right) Z_i^- = V_i^- Z_i^- \tag{29}$$

where

$$V_i^- = \frac{\Delta}{r^5} \left[l(l+1)r - q_j + \frac{4Q^2}{r} \right] \quad (i, j = 1, 2, \; i \neq j) \tag{30}$$

Decoupling by freezing

The gravitational potential when we freeze EM perturbations, $H_1^- = 0$, is

$$V_2^* = \frac{\Delta}{r^5}\left[l(l+1)r - 6M + \frac{4Q^2}{r}\right] \tag{31}$$

V_2^* reduces to the standard RW potential for $Q = 0$. The EM potential freezing metric perturbations, $H_2^- = 0$ becomes

$$V_1^* = \frac{\Delta}{r^5}\left[l(l+1)r + \frac{4Q^2}{r}\right] \tag{32}$$

Similarly to V_2^*, V_1^* reduces to the potential for EM perturbations of Schwarzschild black holes when $Q = 0$. The two potentials can be written in the compact form [Berti and Kokkotas, 2005]

$$V_i^* = \frac{\Delta}{r^5}\left[l(l+1)r + (1-s^2)2M + \frac{4Q^2}{r}\right] \quad (i = |s| = 1, 2) \tag{33}$$

The potential for scalar perturbation of RN black hole reads

$$V_0 = \frac{\Delta}{r^5}\left[l(l+1)r + 2M - \frac{2Q^2}{r}\right] \tag{34}$$

Perturbations in NP formalism

Perturbations of Kerr BH is studied within the Newman-Penrose formalism [Teukolsky, 1973]. This approach works by projecting the spacetime onto the null NP tetrad, and by analysing the geometrical relations between some components of the projected Weyl tensor. All the information of the projected Weyl tensor is contained in five complex scalars. By considering the perturbation of these Weyl scalars, only two of them cannot be set to zero with an infinitesimal gauge transformation. In the perturbed Kerr spacetime, it is convenient to choose the gauge

$$\Psi_1 = \Psi_2 = \Psi_3 = 0 \tag{35}$$

The perturbative degrees of freedomm are classified as

Perturbation Type	Scalar	Vector	Tensor
Spin s	0	+1 or -1	+2 or -2
Teukolsky Function $\psi^{(s)}$	Φ	ϕ_0 or $\bar{\rho}^{*2}\phi_2$	Ψ_0 or $\bar{\rho}^{*4}\Psi_4$

ϕ_0 and ϕ_2 are NP quantities of Maxwell field which describe EM radiation, while Ψ_0 and Ψ_4 are Weyl scalars in NP formalism who can describe gravitational radiation. Here $\bar{\rho} = r + ia\cos\theta$.

Teukolsky equation

By applying the separation of variables to the field with spin s:

$$\psi^{(s)}(t, r\theta, \phi) = \frac{1}{2\pi}\int \sum_{\ell=|s|}^{\infty} \sum_{m=-\ell}^{\ell} e^{-i\omega t + im\phi} S^{(s)}_{\ell m}(\theta) R^{(s)}_{\ell m}(r) d\omega \tag{36}$$

The angular part reads

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{(m+s\cos\theta)^2}{\sin^2\theta} + s + A^{(s)}_{\ell m}\right] S^{(s)}_{\ell m} = 0 \tag{37}$$

where $A^{(s)}_{\ell m}$ is the separation constant and it depends on the frequency ω. The functions $S^{(s)}_{\ell m}$ are known as *spin-weighted spheroidal harmonics*.

The radial equation is

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda^{(s)}_{\ell m}\right] R^{(s)}_{\ell m} = 0 \tag{38}$$

with

$$K = (r^2 + a^2)\omega - am, \quad \lambda^{(s)}_{\ell m} = A^{(s)}_{\ell m} + a^2\omega^2 - 2ma\omega \tag{39}$$

KN linearized perturbation equations

All attempts to decouple the EM and gravitational perturbations of the KN spacetime to date have failed.

In perturbed KN spacetime, it is convenient to choose the "phantom gauge"

$$\phi_0 = \phi_2 = \Psi_2 = 0 \tag{40}$$

The spin weighted fields that capture the gravitational and electromagnetic degrees of freedom are defined as

$$\psi_{-2} = \bar{\rho}^* \Psi_4, \quad \psi_{-1} = \frac{\bar{\rho}^{*3}\Psi_3}{\sqrt{2}}, \quad \psi_1 = \sqrt{2}\bar{\rho}^* \Psi_1, \quad \psi_2 = \Psi_0 \tag{41}$$

The linearized perturbation equations of KN BH are [Chandrasekhar, 1983; Z. Mark etal, 2014]

$$\begin{bmatrix} \mathcal{F}_{\pm 2} + q\mathcal{G}_{\pm 2} & q\delta\mathcal{H}_{\pm 2} \\ q\delta\mathcal{H}_{\pm 1} & \mathcal{F}_{\pm 1} + q\mathcal{G}_{\pm 1} \end{bmatrix} \begin{bmatrix} \psi_{\pm 2} \\ \psi_{\pm 1} \end{bmatrix} = 0 \tag{42}$$

where $q \equiv Q^2/M^2$. The operators $\mathcal{F}_s, \mathcal{G}_s, \delta\mathcal{H}_s$ contain derivatives in both r and θ. To first order in q, the $\delta\mathcal{H}_s$ operator can be neglected.

What we learn so far:

- Decomposing any perturbations around generic static BHs
- 1D eigenvalue problem for static BHs
- Newman-Penrose formalism for Kerr BHs
- 2D eigenvalue problem for Kerr BHs

Open problems in spinning BH:

- NonKerr
- NonGR
- $D > 4$

when we apply the perturbative approach based on NP formalism to generally full rotating backgrounds, we face two issues:

- **matter fields coupled to gravity do not lead to an evident separation of the variables in the perturbation equations;**
- **analytically full rotating solutions are unlikely in alternative theories of gravity.**

Slow-rotation approximation

- Expansion in spherical harmonics
- Couplings to different multipoles
- Only first couplings contribute

The general metric ansatz of slowly-rotating black hole is

$$\bar{ds}^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + C(r)(d\theta^2 + \sin^2\theta d\phi^2) - 2aD(r)\sin^2\theta dt d\phi \tag{43}$$

To first order in a, the radial perturbation equations take the following form [Pani, 2013]

$$\begin{aligned} 0 &= \mathcal{A}_l + \tilde{a}m\bar{\mathcal{A}}_l + \tilde{a}(\mathcal{Q}_l\tilde{\mathcal{P}}_{l-1} + \mathcal{Q}_{l+1}\tilde{\mathcal{P}}_{l+1}) \\ 0 &= \mathcal{P}_l + \tilde{a}m\bar{\mathcal{P}}_l + \tilde{a}(\mathcal{Q}_l\tilde{\mathcal{A}}_{l-1} + \mathcal{Q}_{l+1}\tilde{\mathcal{A}}_{l+1}) \end{aligned} \tag{44}$$

Eigenvalues at first order

The field equations and the boundary conditions are invariant under:

$$
\begin{aligned}
a_{lm} &\to \mp a_{l-m}, \quad & p_{lm} &\to \mp p_{l-m} \\
\tilde{a} &\to -\tilde{a}, & m &\to -m
\end{aligned}
\tag{45}
$$

Therefore, the eigenfrequencies have the schematic form: [Y. Kojima, 1993; Pani etal, 2012]

$$
\omega = \omega_0 + m\omega_1\tilde{a} + \mathcal{O}(\tilde{a}^2) \tag{46}
$$

To first order, only perturbations with the same harmonic index "l" and same parity contribute to the eigenfrequencies

$$
0 = \mathcal{A}_l + \tilde{a}m\bar{\mathcal{A}}_l, \quad 0 = \mathcal{P}_l + \tilde{a}m\bar{\mathcal{P}}_l \tag{47}
$$

NonKerr and NonGR

- accretion disk and jet
- surrounded by dark matters
- stars and exotic compact objects
- alternative theories of gravity

Anisotropic fluid

Without loss of generality, we can assume the slowly-rotating BH is a solution of the effective Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{48}$$

where $T_{\mu\nu}$ is the stress tensor of an anisotropic fluid

$$T_{\mu\nu} = \rho u_\mu u_\nu + P_r k_\mu k_\nu + P_t \Pi_{\mu\nu} \tag{49}$$

where u_μ is the fluid 4-velocity and k_μ is the unit spacelike vector orthogonal to u_μ, i.e.

$$-u^\mu u_\mu = k^\mu k_\mu = 1, \quad u^\mu k_\mu = 0 \tag{50}$$

and

$$\Pi_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu \tag{51}$$

is a projection operator onto a two-surface orthogonal u^μ and k^μ, i.e.

$$u_\mu \Pi^{\mu\nu} V_\nu = k_\mu \Pi^{\mu\nu} V_\nu = 0 \tag{52}$$

for any vector V^μ.

Perturbations of the stress tensor

The perturbations of the stress tensor can be expressed as

$$\delta T^{\mu\nu} = (\rho + P_t)\delta(u^\mu u^\nu) + (P_r - P_t)\delta(k^\mu k^\nu) + P_t \delta g^{\mu\nu}$$
$$+(\delta\rho + \delta P_t)u^\mu u^\nu + (\delta P_r - \delta P_t)k^\mu k^\nu + \delta P_t g^{\mu\nu} \quad (53)$$

The perturbations δu^μ of fluid's 4-velocity are described by three variables R, U, V. The perturbations δk^μ are described by two variables Q, P.

- polar: $R, V, P, \delta\rho, \delta P_r, \delta P_t$
- axial: U, Q

For simplicity, we only consider axial sector. In tetrad formalism, the effective Einstein equation is

$$R_{\bar{a}\bar{b}} = 8\pi T^t_{\bar{a}\bar{b}} \quad (54)$$

where

$$T^t_{\bar{a}\bar{b}} \equiv \left(T_{\bar{a}\bar{b}} - \frac{1}{2} T \eta_{\bar{a}\bar{b}}\right) \quad (55)$$

The perturbation equation is given by

$$\mathcal{E}_{\bar{a}\bar{b}} \equiv \delta R_{\bar{a}\bar{b}} - 8\pi \delta T^t_{\bar{a}\bar{b}} = 0 \quad (56)$$

We obtain the components of perturbed stress tensor

$$\delta T^{t}_{\bar{a}\bar{b}} = \begin{pmatrix} \frac{2a\Omega\kappa(U+h_0)S_\phi}{A} & -\frac{a\Omega\sqrt{B}\sigma(Q+h_1)S_\phi}{\sqrt{A}} & -\frac{\sigma(U+h_0)S_\theta}{\sqrt{AC}} & -\frac{\sigma(U+h_0)S_\phi}{\sqrt{AC}\sin\theta} \\ * & 0 & \frac{\sqrt{B}\sigma(Q+h_1)S_\theta}{\sqrt{C}} & \frac{\sqrt{B}\sigma(Q+h_1)S_\phi}{\sqrt{C}\sin\theta} \\ * & * & 0 & \frac{a\Omega\kappa(U+h_0)\sin\theta S_\theta}{A} \\ * & * & * & \frac{2a\Omega\kappa(U+h_0)S_\phi}{A} \end{pmatrix} \tag{57}$$

where we have defined $\kappa = \rho + P_t$ and $\sigma = P_r - P_t$. We can see from the perturbed stress tensor that $Q^{\ell m} + h_1^{\ell m}$ corresponds to the momentum flow. Since the fluid we consider has no friction, we set

$$Q^{lm} + h_1^{lm} = 0 \tag{58}$$

for the perturbation of stress tensor. Eventually, We have three equations for the axial perturbations $h_0^{\ell m}, h_1^{\ell m}, U^{lm}$

$$P_0 = J\beta^0_{\ell m} + ima[(J-2)\chi^0_{\ell m} + \tilde{\alpha}^0_{\ell m} + \eta^0_{\ell m}] = 0 \tag{59}$$

$$P_1 = J\beta^1_{\ell m} + ima[(J-2)\chi^1_{\ell m} + \tilde{\alpha}^1_{\ell m} + \eta^1_{\ell m}] = 0 \tag{60}$$

$$P_2 = Jt_{\ell m} + imag_{\ell m} = 0 \tag{61}$$

Since we consider only the slowly-rotating case, i.e. the spin parameter a contributes a linear correction to the static case, we can assume the master variable has the following form

$$\Psi = \sqrt{\frac{AB}{C}}(h_1 + maf_{h_1}) \tag{62}$$

Then we can also obtain a Schrodinger-like wave equation

$$\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V^{(0)} - maV^{(a)})\Psi = 0 \tag{63}$$

We find that the corrective function f_{h_1} is

$$f_{h_1} = c + \frac{1}{\omega}\frac{D}{C} \tag{64}$$

where c is a constant and the potentials are

$$\begin{aligned} V^{(0)} &= (J-2)\frac{A}{C} + \sqrt{ABC}\left(\sqrt{AB}\left(\frac{1}{\sqrt{C}}\right)'\right)' \\ V^{(a)} &= 2\omega\frac{D}{C} + \frac{1}{J\omega}\left[\sqrt{AB}\left(\sqrt{AB}\left(\frac{D}{C}\right)'\right)' - AB\frac{C'}{C}\left(\frac{D}{C}\right)'\right] \end{aligned} \tag{65}$$

Slow-rotation approximation

- Matter perturbations decoupled from gravitational perturbations for axial sector
- Modified Regge-Wheeler equation
- Any metric, Any theory

We attribute parameterized deviations from GR to a phenomenological stress tensor. Our results are applicable when the dynamics of matter fields or high-dimensional operators can be neglected.

Potentials of KN gravitational perturbation

For KN BH, to first order in spin a, we have

$$A(r) = B(r) = \Delta/r^2, \quad C(r) = r^2, \quad D(r) = \frac{2M}{r} - \frac{Q^2}{r^2} \tag{66}$$

where $\Delta = r^2 - 2Mr + Q^2$, the potentials reduce to

$$V^{(0)} = \frac{\Delta}{r^5}\left[Jr - 6M + \frac{4Q^2}{r}\right] = V_2^*$$
$$V^{(a)} = \frac{8\Delta\left(3Mr^2(3r-7M) + 7Q^2r(4M-r) - 9Q^4\right)}{J\omega r^{10}} + \frac{2\omega\left(2Mr - Q^2\right)}{r^4} \tag{67}$$

These results suppose to give the gravitational QNMs of slowly-rotating and weakly-charged KN BH (**Freezing the perturbations of electromagnetic field**).

Einsteinian cubic gravity

The action of Einsteinian cubic gravity is [Bueno and Cano, 2016]

$$S = \frac{1}{16\pi G} \int dx^4 (R + \lambda P) \tag{68}$$

where the cubic curvature term is

$$\mathcal{P} = 12 R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} R_{\rho}{}^{\gamma}{}_{\sigma}{}^{\delta} R_{\gamma}{}^{\mu}{}_{\delta}{}^{\nu} + R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\gamma\delta} R_{\gamma\delta}^{\mu\nu} - 12 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 8 R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu} \,. \tag{69}$$

The spherically symmetric BH up to first order in coupling ϵ is

$$f(r) = 1 - \frac{2M}{r} + \epsilon \left(\frac{216 M^6}{r^6} - \frac{368 M^7}{r^7} \right) + \mathcal{O}(\epsilon^2) \tag{70}$$

where we have introduce a dimensionless coupling constant ϵ by setting

$$\lambda = M^4 \epsilon \tag{71}$$

Master wave equations

When the coupling parameters ϵ are taken to be perturbatively small, one can use the zeroth-order GR equations to reduce the problem to second order differential equations

$$\frac{d^2\Psi^{\pm}}{dr_*^2} + (\omega^2 - V_{GR}^{\pm} - \epsilon V_P^{\pm})\Psi^{\pm} = 0 \tag{72}$$

where the corrected potentials are

$$V_P^- = \frac{8M^5\left[84984M^3 + 2(2875J - 57483)M^2 r - 9(599J - 5705)Mr^2 + 1260(J-6)r^3\right]}{r^{10}} + \frac{2016M^5 w^2(2M-r)}{r^6} \tag{73}$$

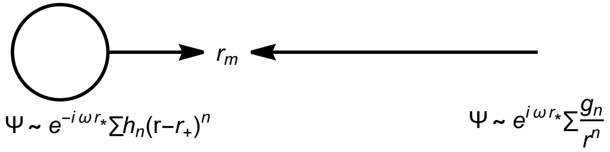
$$V_P^+ = \cdots \tag{74}$$

The corrected potential from $V^{(0)}$ is

$$\tilde{V}_P^- = \frac{8M^6[1104M^2 - 2(504+23J)Mr + 27(8+J)r^2]}{r^{10}} \tag{75}$$

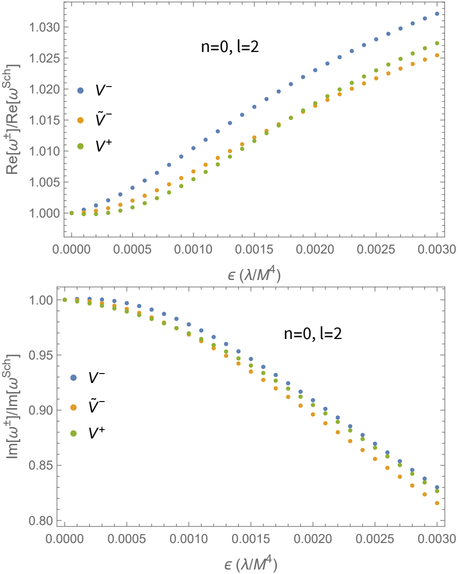
Shooting method

We perform two integrations: (I) from the horizon to a matching point $r = r_m$ imposing the purely ingoing wave boundary condition, (II) from infinity to the matching point, imposing no incoming waves at infinity. Then, we construct a linear combination of the two solutions and fix the ratio of the coefficients in order for the linear combination to be C^0 at the matching point. However, for a generic frequency the linear combination will have discontinous derivatives. Finally, we find the eigenvalue by imposing that the linear combination is C^1.



The expansion coefficients h_n, g_n can be determined order by order.

QNM spectrum



Conclusions

- The effect of the small spin on BH linear perturbations is just to modify the Regge-Wheeler potential.
- The modified potential gives the gravitational QNMs when we can neglect the dynamics of matter fields or high-dimensional operators.

Discussions

- Generalizing to arbitrary order in slow-rotation approximation
- Developing more versatile semianalytical methods
- Metric reconstruction: connections between metric perturbations and curvature perturbations