Analytical methods for cosmological correlators

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Based on:

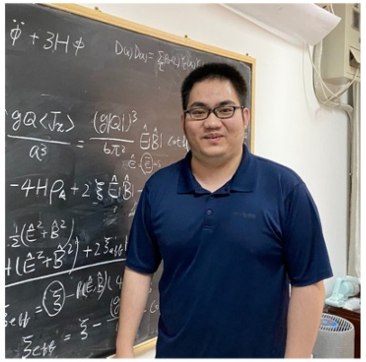
w/ Zhehan Qin: **[Zhehan's poster]**

1. JHEP **10** (2022) 192 [2205.01692]
2. JHEP **04** (2023) 059 [2208.13790]
3. JHEP **07** (2023) 001 [2301.07047]
4. JHEP **09** (2023) 116 [2304.13295]
5. JHEP **01** (2024) 168 [2308.14802]
6. w/ Hongyu Zhang:
JHEP **04** (2023) 103 [2211.03810]
7. w/ Jiaju Zang:
JHEP **03** (2024) 070 [2309.10849]

8: w/ Bingchu Fan: 2403.07050,

9. w/ Bingchu Fan & J. Zang: 24XX.XXXXX

10. w/ Haoyuan Liu & Z. Qin: 24XX.XXXXX



Bingchu Fan
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Jiaju Zang
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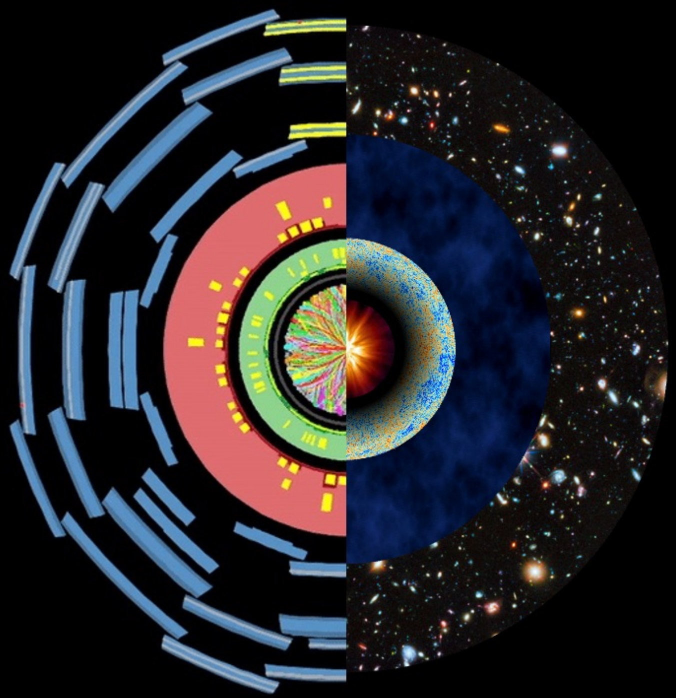
Hongyu Zhang
张洪语

Outline

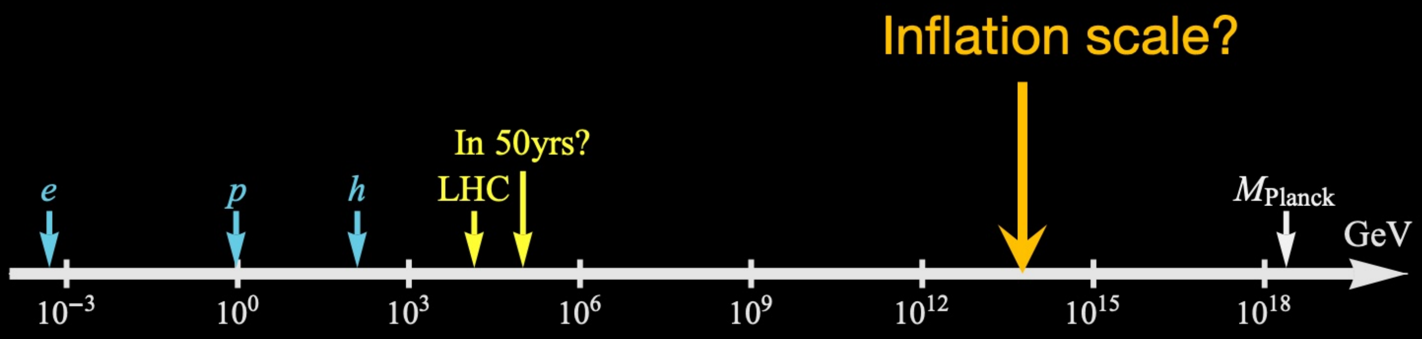
1. Background [Cosmo collider signal <-> (non)analyticity of correlators]
2. Cosmological correlators [general structure | partial Mellin-Barnes]
3. Loops: Analytical techniques [factorization | spectral | dispersion]
4. Nested time integrals: family tree decomposition
5. Conformal amplitudes in FRW [general rule | energy int | inflation limit]

The cosmological collider program



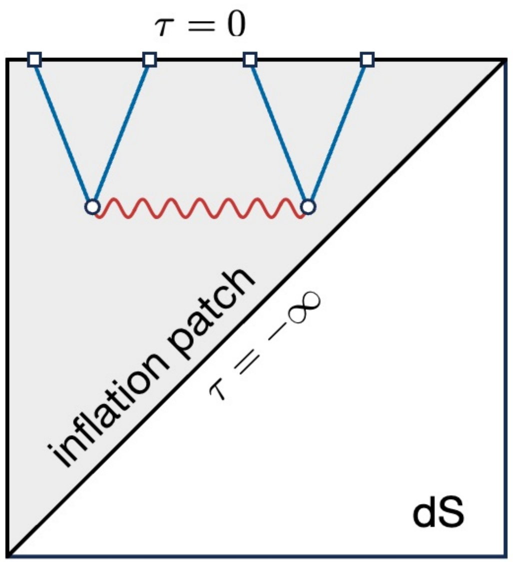
The primordial universe: very high energy scale

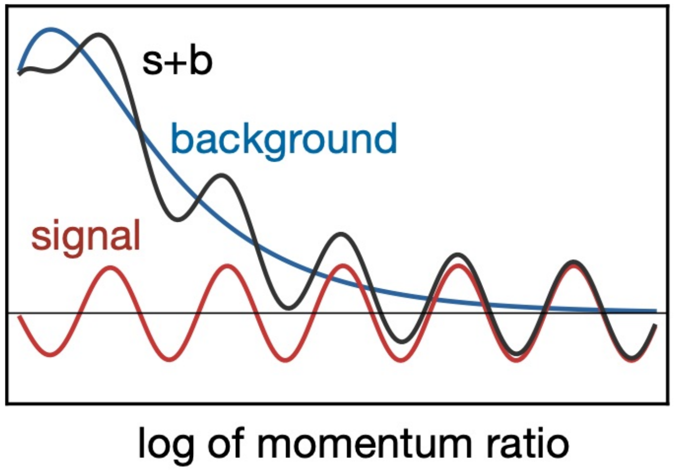
Quantum fluctuations => large-scale structure
A unique window to fundamental physics at inflation scale

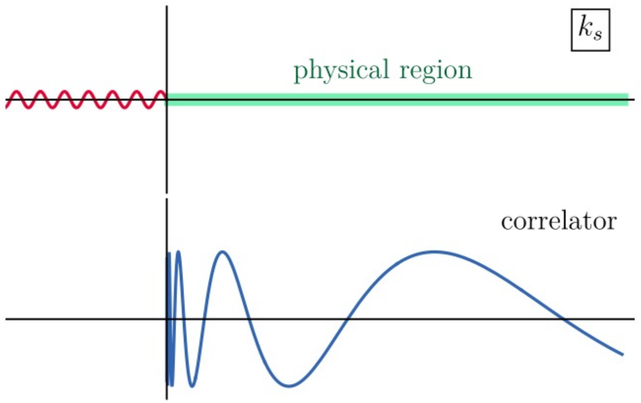


[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]

Cosmological collider signal







$$\mathcal{T} \sim \mathcal{S}(k_s) k_s^{\pm i\tilde{\nu}} + \mathcal{B}(k_s)$$

Promising observables

[Sohn et al., 2404.07203
Cabass et al., 2404.01894]

2~4 orders in the future

Rich physics

particle mass / spin / int. etc

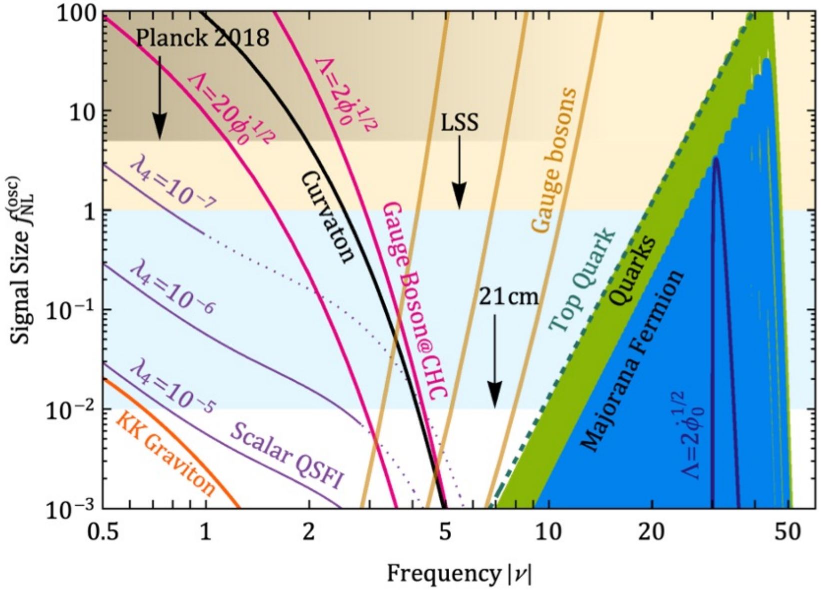
expansion history

Signals ~ nonanalyticity

Branch cut / factorization / cutting rule / OPE

[Qin, **ZX**, 2304.13295; 2308.14802]

Phenomenological motivations



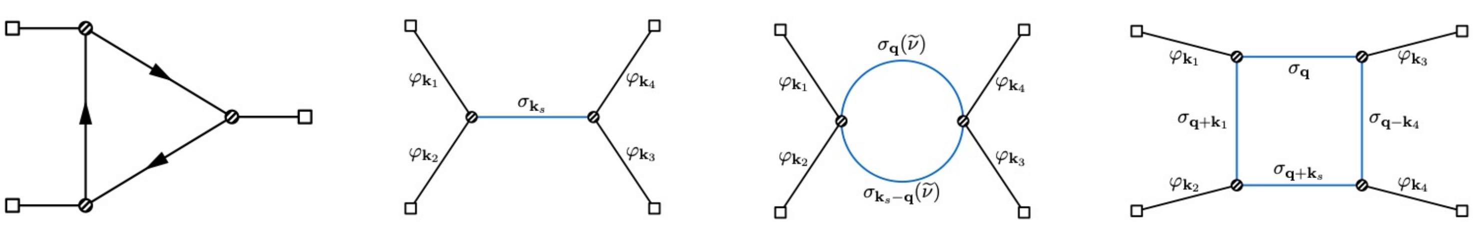
[Lian-Tao Wang, **ZX**, 1910.12876]

Over the years, many particle models identified in SM/BSM, with large signals

Many types of diagrams (tree + loop) involved

Understanding the amplitudes!

- efficient numerical implementation
- analytical structure



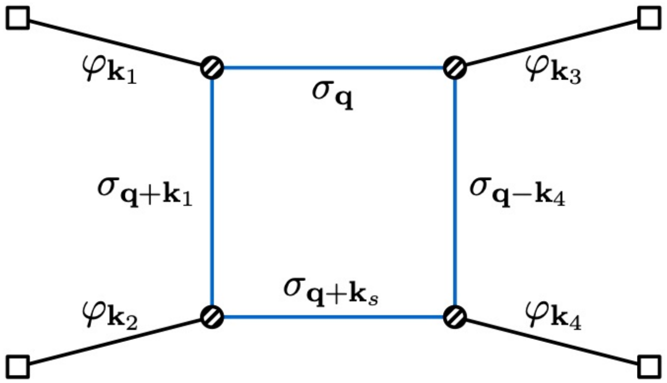
Cosmological correlators: general structure

[See Chen, Wang, **ZX**, 1703.10166 for a review]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\widetilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

vertex int loop int ext line bulk line

Complications and strategies



Special functions in propagators

☐ partial Mellin-Barnes [Qin, **ZX**, 2205.01692, 2208.13790 etc.]

Loop (momentum) integral

☐ spectral [**ZX**, Zhang, arXiv:2211.03810] ☐ dispersion [ongoing]

Nested time integral

☐ family tree [**ZX**, Zang, 2309.10849]

Other methods: bootstrap [Arkani-Hamed et al. 1811.00024 etc.]
AdS + Mellin [Sleight 1907.01143 etc.] diff eq [Arkani-Hamed et al. 2312.05303]

Partial Mellin-Barnes representation

[Qin, **ZX**, 2205.01692, 2208.13790]

Mellin transform & Mellin-Barnes rep:

$$F(s) = \int_0^\infty \mathrm{d}x\, x^{s-1} f(x) \qquad\qquad f(x) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, x^{-s} F(s)$$

Expanding in dilatation eigenmode [dS counterpart of Fourier transform in flat space]

Partial Mellin-Barnes rep: MB rep for all bulk lines; Special functions => powers

For example: Massive scalar propagator [Hankel function]

$$\mathrm{H}_\nu^{(1)}(-k\tau) = \frac{1}{\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \left(\frac{k}{2}\right)^{-2s} (-\tau)^{-2s} e^{(2s-\nu-1)\pi\mathrm{i}/2}\, \Gamma\left[s-\frac{\nu}{2}, s+\frac{\nu}{2}\right]$$

Time and momentum factorized

All time and momentum integrals factorized; We can deal with them separately:

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d \boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\right]$$

bulk lines — loop int — nested time int

bulk lines, loop int → Left poles

loop int, nested time int → Right poles

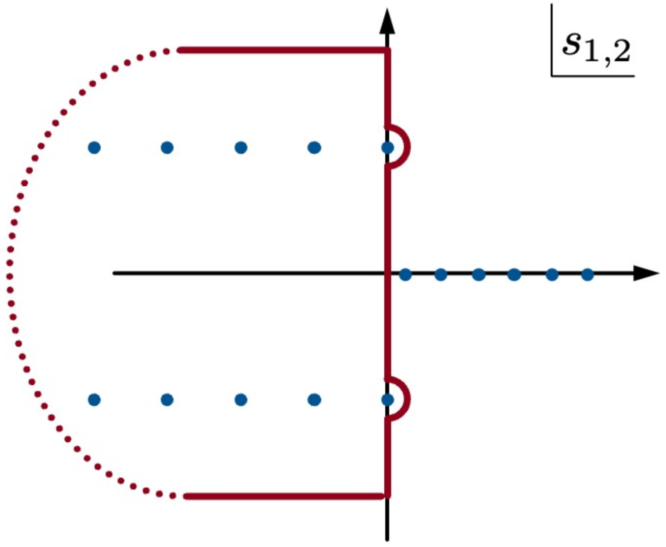
Loop integrals similar to flat space [simple loops doable]

Time integrals more challenging: arbitrary time orderings

Mellin integrands typically meromorphic [only poles]

Final results by residue theorem: pole collecting

Pole structure encodes rich physics!



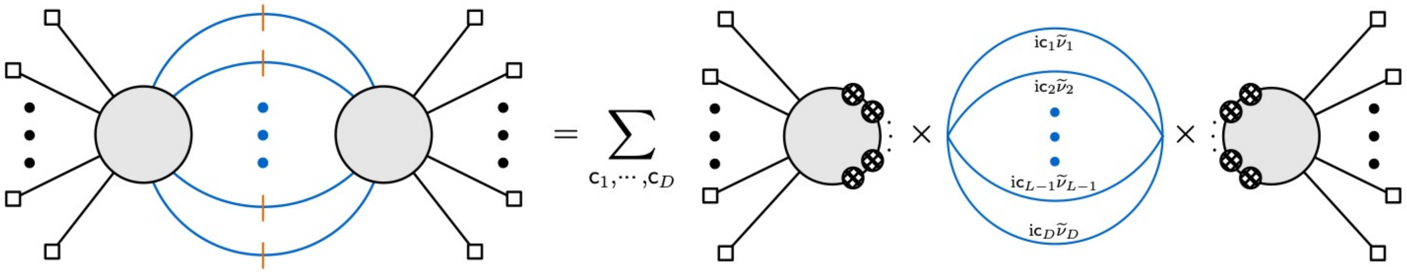
All-loop factorization theorem & cutting rule

[Zhehan Qin, **ZX**, 2304.13295; 2308.14802]

Thanks to PMB, nonanalyticities in K all come from the loop integral:

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d \boldsymbol{q} K(\boldsymbol{q}, \boldsymbol{k})^\alpha\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\right]$$

Detailed analysis of loops shows that all (nonlocal) signals are factorized & cuttable



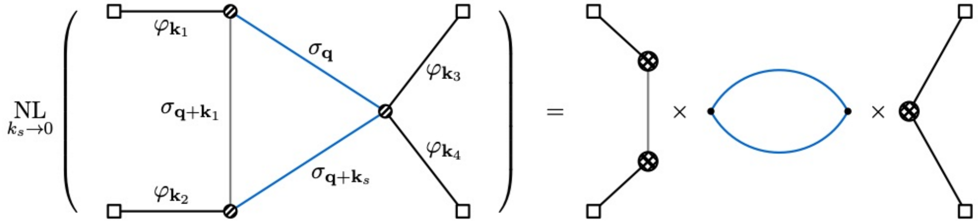
Moreover, all loop signals are analytically calculable:

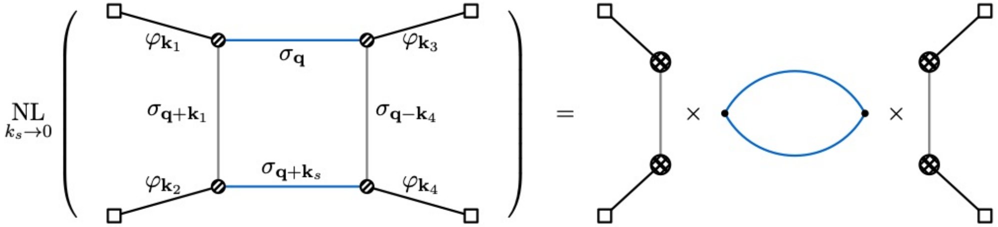
$$\mathfrak{M}_{\mathsf{c}_1\cdots\mathsf{c}_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}}\Gamma\left[\begin{matrix} -\sum_{i=1}^{D}\mathsf{c}_i\mathrm{i}\widetilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^{D}\mathsf{c}_i\mathrm{i}\widetilde{\nu}_i \end{matrix}\right]\prod_{\ell=1}^{D}\left\{\Gamma\left[\tfrac{3}{2}+\mathsf{c}_\ell\mathrm{i}\widetilde{\nu}_\ell, -\mathsf{c}_\ell\mathrm{i}\widetilde{\nu}_\ell\right]\left(\frac{P}{2}\right)^{2\mathrm{i}\mathsf{c}_\ell\widetilde{\nu}_\ell}\right\}$$

Application: signals from arbitrary 1-loop graphs

[Zhehan Qin, **ZX**, 2301.07047; 2304.13295]

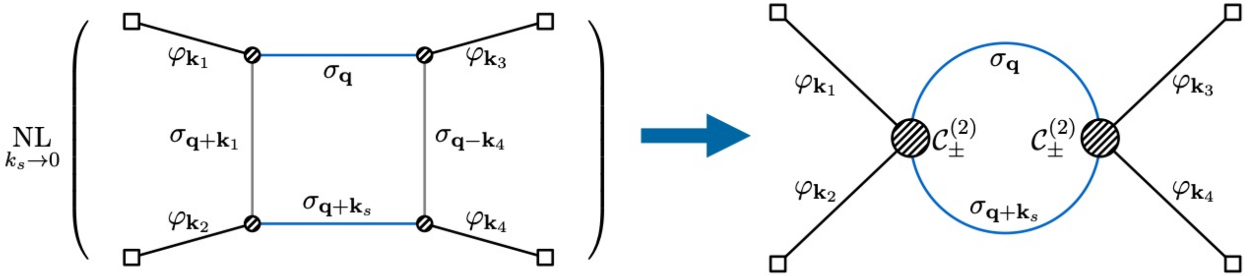
The factorization enables precise determination of leading order nonlocal signals. E.g.:

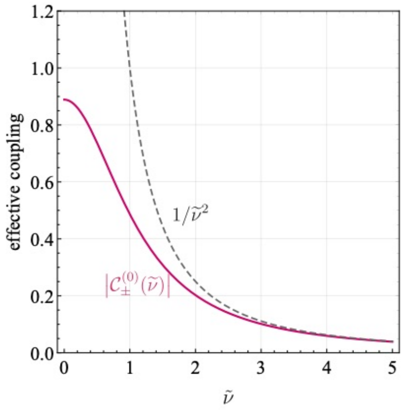




Subgraphs computable in closed form with improved bootstrap equations [2301.07047]

Upshot: Cut the soft lines and pinch the hard lines (OPE; pinched coupling) [2304.13295]



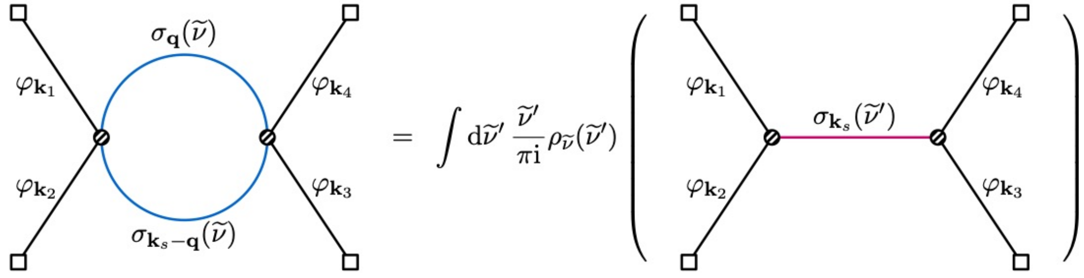


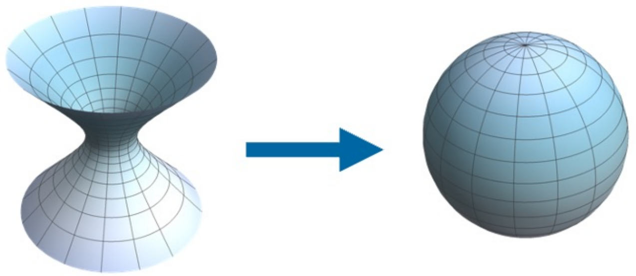
The effective coupling as a function of intermediate mass

Bootstrap by spectral decomposition

[**ZX**, Hongyu Zhang, 2211.03810]

Loops greatly simplified with new strategies in certain cases: spectral decomposition

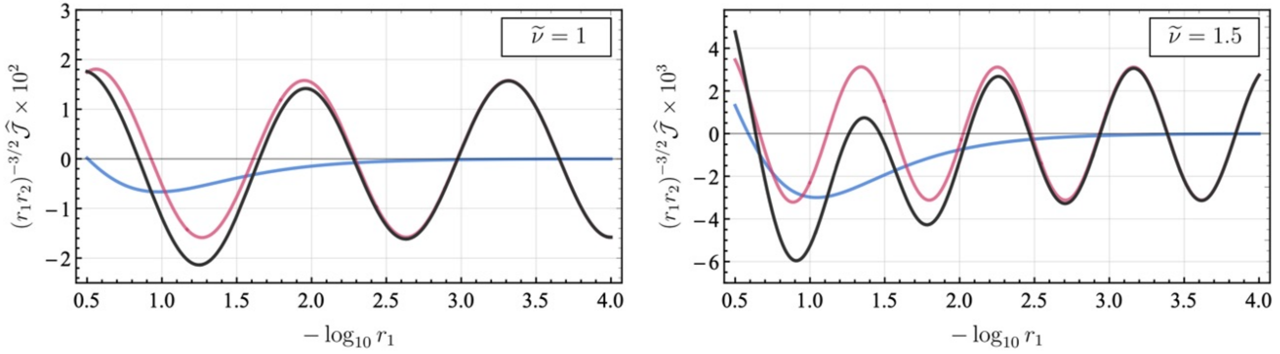




Rewrite bubble 1-loop as linear superposition of tree graphs with all possible masses.

The spectral density obtainable by Wick-rotating dS to sphere or AdS

With spectral method, we get the first and hitherto only known complete analytical result for massive 1-loop processes

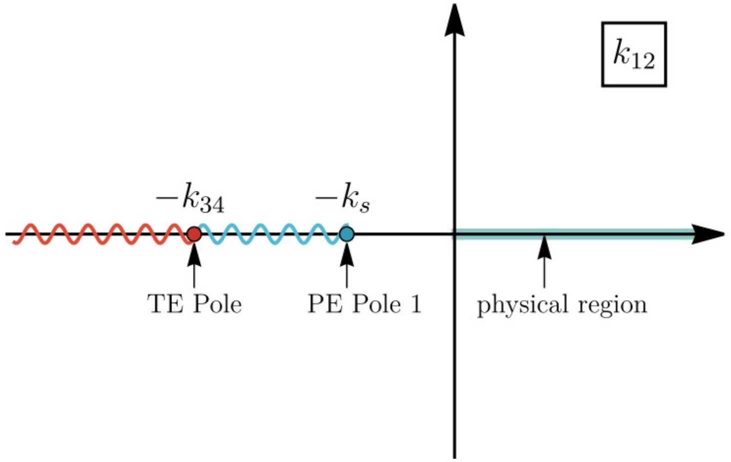


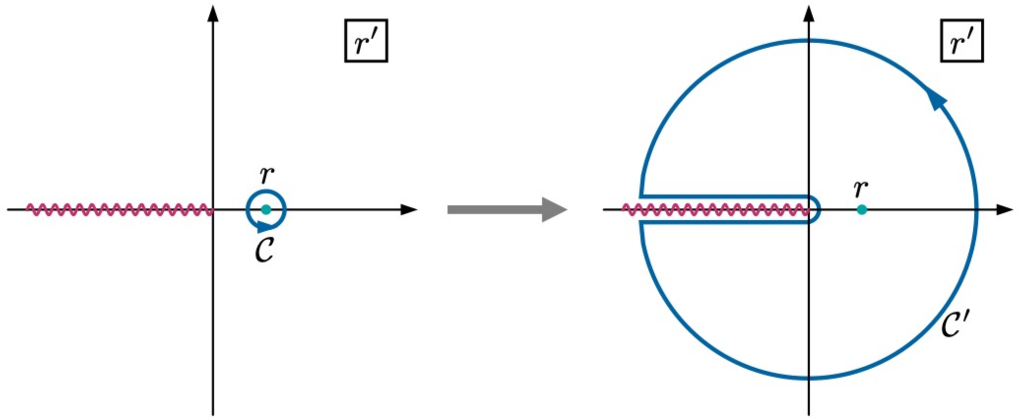
Bootstrap by dispersion relations

[Haoyuan Liu, Zhehan Qin, **ZX**, 240X.XXXXX]

The study of analyticity allows us to locate all singularities on the complex plane

=> Bootstraping complex graphs by gluing simpler ones. The glue: dispersion integral





Dispersion integrals are insensitive to UV (local) physics

New and much simplified analytical expression for loops; UV and IR neatly separated

In particular: we identify an "irreducible background" demanded by analyticity

Family tree decomposition

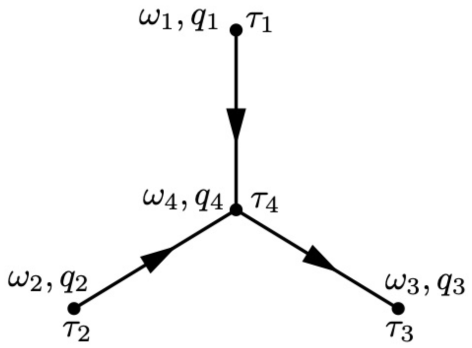
[**ZX**, Zang, 2309.10849]

$$\mathcal{T}\big(\{\boldsymbol{k}\}\big) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\right]$$

bulk lines — loop int — nested time int

The most general time integral: $$(-\mathrm{i})^N \int_{-\infty}^0 \prod_{\ell=1}^N \left[\mathrm{d}\tau_\ell\,(-\tau_\ell)^{q_\ell-1} e^{\mathrm{i}\omega_\ell\tau_\ell}\right] \prod \theta(\tau_j - \tau_i)$$

It naturally acquires a graphic representation [NOT original Feynman diagrams]:



$$= (-\mathrm{i})^4 \int \prod_{\ell=1}^4 \left[\mathrm{d}\tau_\ell\,(-\tau_\ell)^{q_\ell-1} e^{\mathrm{i}\omega_\ell\tau_\ell}\right]\theta(\tau_4-\tau_1)\theta(\tau_4-\tau_2)\theta(\tau_3-\tau_4)$$

Family tree decomposition

Complications all from theta functions

Irremovable, but can flip directions, at the expense of additional factorized graphs

$$\theta(\tau_1 - \tau_2) + \theta(\tau_2 - \tau_1) = 1$$

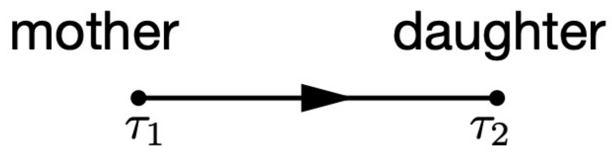
Family tree decomposition:

We always flip the directions such that all nested graphs are partially ordered

Partial order:

A mother can have any number of daughters

but a daughter must have only one mother



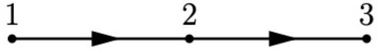
Every resulting nested graph can be interpreted as a maternal family tree

A useful notation for family trees: $\big[12(34\cdots)(5\cdots)\big]$

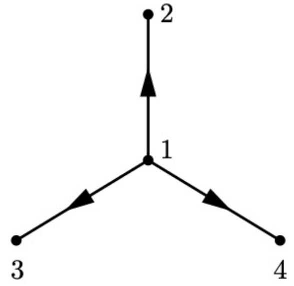
sisters

mother-daughter

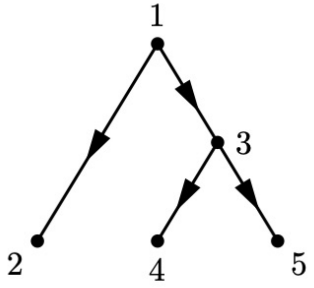
Examples:



$$[123] = (-\mathrm{i})^3 \int \prod_{i=1}^{3} \Big[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \Big] \theta_{32}\theta_{21}$$



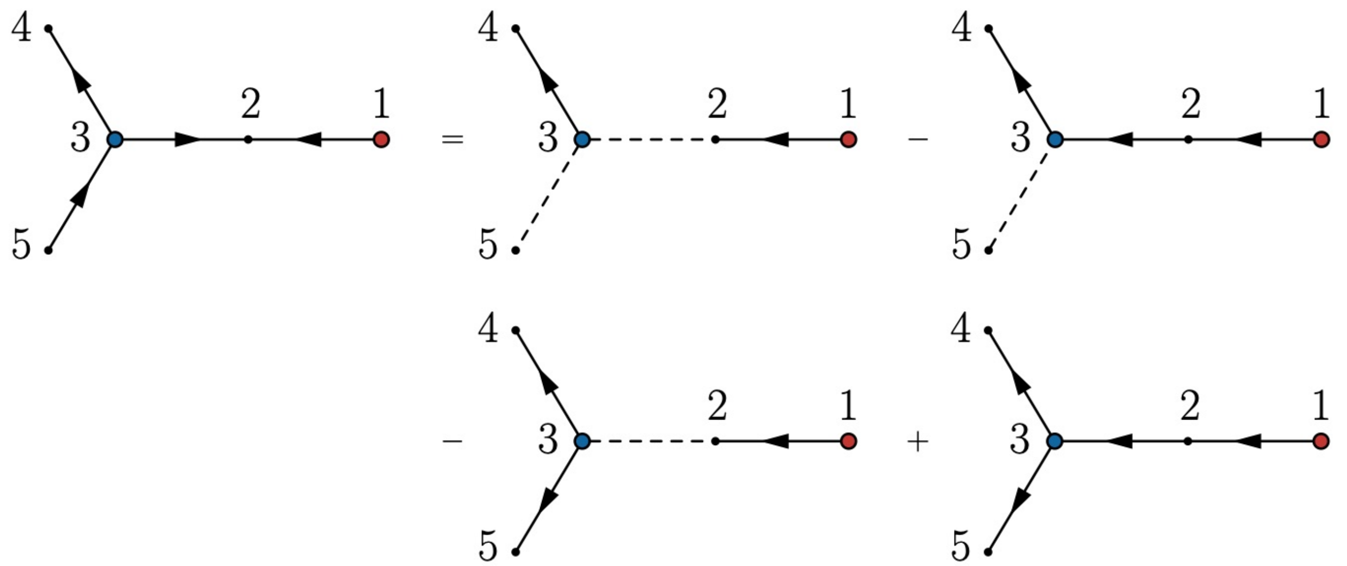
$$[1(2)(3)(4)] = (-\mathrm{i})^4 \int \prod_{i=1}^{4} \Big[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \Big] \theta_{41}\theta_{31}\theta_{21}$$



$$\big[1(2)\big(3(4)(5)\big)\big] = (-\mathrm{i})^5 \int \prod_{i=1}^{5} \Big[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \Big] \theta_{43}\theta_{53}\theta_{31}\theta_{21}$$

$$\theta_{ij} \equiv \theta(\tau_i - \tau_j)$$

Example of family tree decomposition



Choose Site 1 as the earliest

1->2 good   2->3 flip

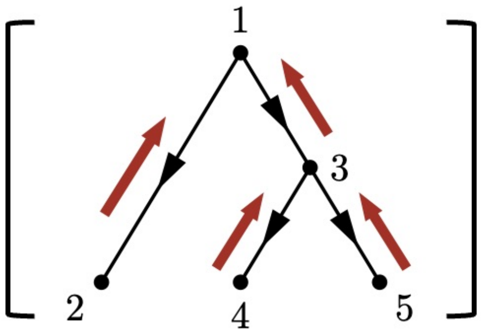
3->4 good   3->5 flip

$$\int \prod_{\ell=1}^{N} \left[\mathrm{d}\tau_\ell \, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell} \right] \theta(\tau_2 - \tau_1)\theta(\tau_2 - \tau_3)\theta(\tau_4 - \tau_3)(\tau_3 - \tau_5)$$

$$= [12]\,[34]\,[5] - [1234]\,[5] - [12]\,[3(4)(5)] + [123(4)(5)]$$

Computing the family tree

Expand-and-integrate strategy works, but more streamlined with MB reps

$\mathcal{R}$ 

→ $\displaystyle\int_{-\infty}^{\tau_1} \mathrm{d}\tau_2\,(-\tau_2)^{q_2-1} e^{\mathrm{i}\omega_2\tau_2} = (-\tau_1)^{q_2}\mathrm{E}_{1-q_2}(-\mathrm{i}\omega_2\tau_1)$

(exp int)

→ $\displaystyle\mathrm{E}_p(z) = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\Gamma(s) z^{-s}}{s+p-1}$

MB rep

→ next layer: again powers and exp → go through all layers → finish MB int

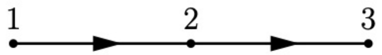
Mellin integrals finished by the residue theorem, with a series expansion:

$$[\mathscr{P}(\widehat{1}2\cdots N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}} \sum_{n_2,\cdots,n_N=0}^{\infty} \Gamma(q_{1\cdots N} + n_{2\cdots N}) \prod_{j=2}^{N} \frac{(-\omega_j/\omega_1)^{n_j}}{(\widetilde{q}_j + \widetilde{n}_j) n_j!}$$

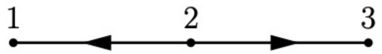
earliest site

sum of all q's on Site j and her descendants $(q_{12\cdots} \equiv q_1 + q_2 + \cdots)$

Examples:



$$[123] = \frac{\mathrm{i}}{(\mathrm{i}\omega_1)^{q_{123}}} \sum_{n_2,n_3=0}^{\infty} \frac{(-1)^{n_{23}}\Gamma[n_{23}+q_{123}]}{n_2!n_3!(n_{23}+q_{23})(n_3+q_3)} \left(\frac{\omega_2}{\omega_1}\right)^{n_2} \left(\frac{\omega_3}{\omega_1}\right)^{n_3}$$



$$[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}} \sum_{n_1,n_3=0}^{\infty} \frac{(-1)^{n_{13}}\Gamma[n_{13}+q_{123}]}{n_1!n_3!(n_1+q_1)(n_3+q_3)} \left(\frac{\omega_1}{\omega_2}\right)^{n_1} \left(\frac{\omega_3}{\omega_2}\right)^{n_3}$$

All family trees are multivariate hypergeometric series

Always expanded in reciprocal of earliest energy, prefactor gives the monodromy

When do FTD, always ask the maximal energy to sits at the earliest site

For simple family trees, the series sum to named hypergeometric functions [all dressed]

$$[1] = \frac{-\mathrm{i}}{(\mathrm{i}\omega_1)^{q_1}}\Gamma[q_1]$$ Euler Gamma function

$$[12] = \frac{-1}{(\mathrm{i}\omega_1)^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right]$$ Gauss hypergeometric function

$$[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$ Appell function

$$[123] = \frac{\mathrm{i}}{(\mathrm{i}\omega_1)^{q_{123}}}\ {}^{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle|\begin{matrix} \text{-}\ , q_3 \\ \text{-}\ , q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right]$$ Kampé de Fériet function

$$[1(2)\cdots(N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}}\mathcal{F}_A\left[q_{1\cdots N}\middle|\begin{matrix} q_2, \cdots, q_N \\ q_2+1, \cdots, q_N+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, \cdots, -\frac{\omega_N}{\omega_1}\right]$$ Lauricella function

… while more complicated family trees are not yet named

Flexibility, functional identities, analytical continuation

[Bingchu Fan, **ZX**, Jiaju Zang, to appear]

The flexibility of MB rep leads to many distinct expansions of family trees in terms of large / small single energy, partial energy, total energy, or energy differences.

The many expansions of the same function yield many functional identities when the family tree sums to known functions:

$[12] = [12]$

$$\frac{1}{\omega_1^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_2, q_{12}\\ q_2+1\end{matrix}\middle|\, -\frac{\omega_2}{\omega_1}\right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}1, q_{12}\\ q_2+1\end{matrix}\middle|\, \frac{\omega_2}{\omega_{12}}\right]$$

$[12] + [21] = [1]\,[2]$

$$\frac{1}{\omega_1^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_2, q_{12}\\ q_2+1\end{matrix}\middle|\, -\frac{\omega_2}{\omega_1}\right] + \frac{1}{\omega_2^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_1, q_{12}\\ q_1+1\end{matrix}\middle|\, -\frac{\omega_1}{\omega_2}\right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1}\omega_2^{q_2}}$$

$[123] + [2(1)(3)] = [1]\,[23]$

$$\frac{1}{\omega_1^{q_{123}}}\,{}^{2+1}\mathcal{F}_{1+1}\left[\begin{matrix}q_{123}, q_{23}\\ q_{23}+1\end{matrix}\middle|\begin{matrix}\text{-}\,, q_3\\ \text{-}\,, q_3+1\end{matrix}\middle|\, -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right] + \frac{1}{\omega_2^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix}q_1, q_3\\ q_1+1, q_3+1\end{matrix}\middle|\, -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$

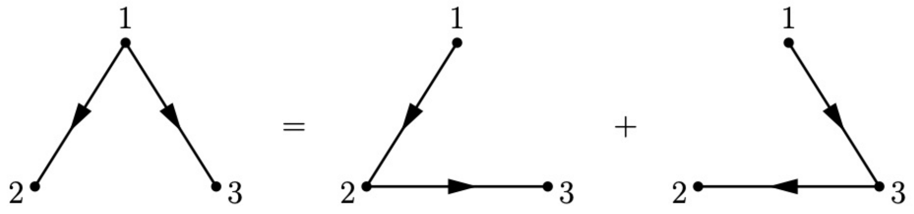
$$= \frac{\Gamma[q_1]}{\omega_1^{q_1}\omega_2^{q_{23}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_3, q_{32}\\ q_3+1\end{matrix}\middle|\, -\frac{\omega_3}{\omega_2}\right]$$

More importantly, when series do not close, the identities amount to analytical continuation beyond the region of convergence

Minimal set of functions: family chains

birthday rule: Compare the birthdays of all family members and sum over all possibilities

$$\theta_{21}\theta_{31}(\theta_{32}+\theta_{23}) = \theta_{32}\theta_{21} + \theta_{23}\theta_{31}$$

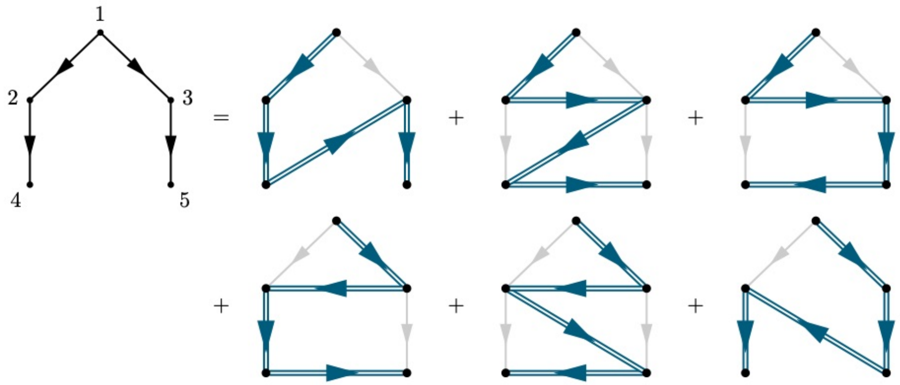


Formally, take shuffle products recursively among all subfamilies

---

Example:

$$\begin{aligned}\big[1(24)(35)\big] &= \big\{1(24) ⧢ (35)\big\} \\ &= \{12435\} + \{12345\} + \{12354\} \\ &+ \{13245\} + \{13254\} + \{13524\}\end{aligned}$$



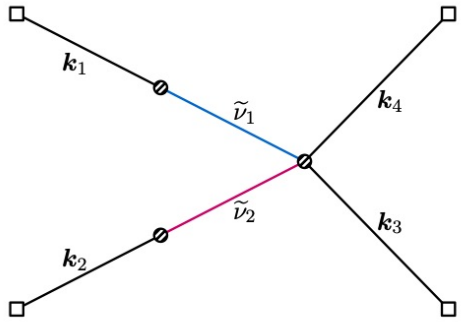
Family trees over-complete: further decomposable to chains; tree topology erased

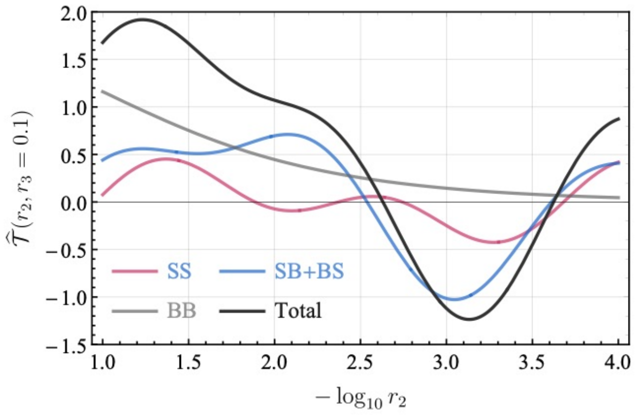
Family chains: iterated integrals; Hopf algebra; transcendental weight; symbology?

Applications

1. Cosmological collider physics with multiple massive exchanges [**ZX**, Zang, 2309.10849]
[See also Aoki et al.: 2404.09547]

Partial MB + family tree analytical expressions enables fast numerical implementation





2. Conformal amplitudes in arbitrary power-law FRW universe [Fan, **ZX**, 2403.07050]

$$S[\phi_c] = -\int \mathrm{d}^{d+1}x\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi_c)^2 + \frac{1}{2}\xi R\phi_c^2 + \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi_c^n\right] \qquad \xi \equiv (d-1)/(4d)$$

An important class of toy model; rich structure [cosmo polytope; canonical form; symbology]

Recent works explored the diff eqs [Arkani-Hamed et al. 2312.05303 etc.]

With family trees, we found full analytical answers

Conformal amplitudes in FRW

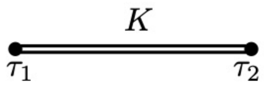
[Fan, **ZX**, 2403.07050]

Two types of "amplitudes": correlators & wavefunction (coefficients)

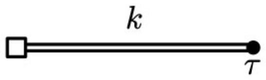
Similar structures but distinct physical meanings

For wavefunction: [correlators similar, see 2403.07050]

bulk line: $\widetilde{G}(K;\tau_1,\tau_2) = e^{-\mathrm{i}K(\tau_1-\tau_2)}\theta_{12} + e^{+\mathrm{i}K(\tau_1-\tau_2)}\theta_{21} - e^{+\mathrm{i}K(\tau_1+\tau_2)}$

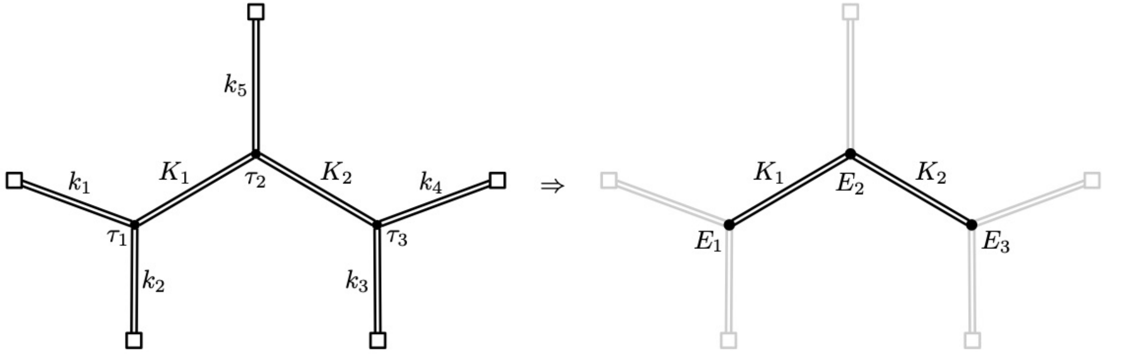


boundary line: $B(k;\tau) = e^{\mathrm{i}k\tau}$



For each vertex, only total external energy relevant: $B(k_1;\tau)B(k_2;\tau) = B(k_{12};\tau)$

A tree graph fully determined by its external energies E (and power q) at all sites, and internal energies K on all bulk lines



$E_1 = k_{12}, \quad E_2 = k_5, \quad E_3 = k_{34}$

Tree conformal amplitudes ~ sum & products of family trees [finally, like in flat space!]

Rule for wavefunctions:

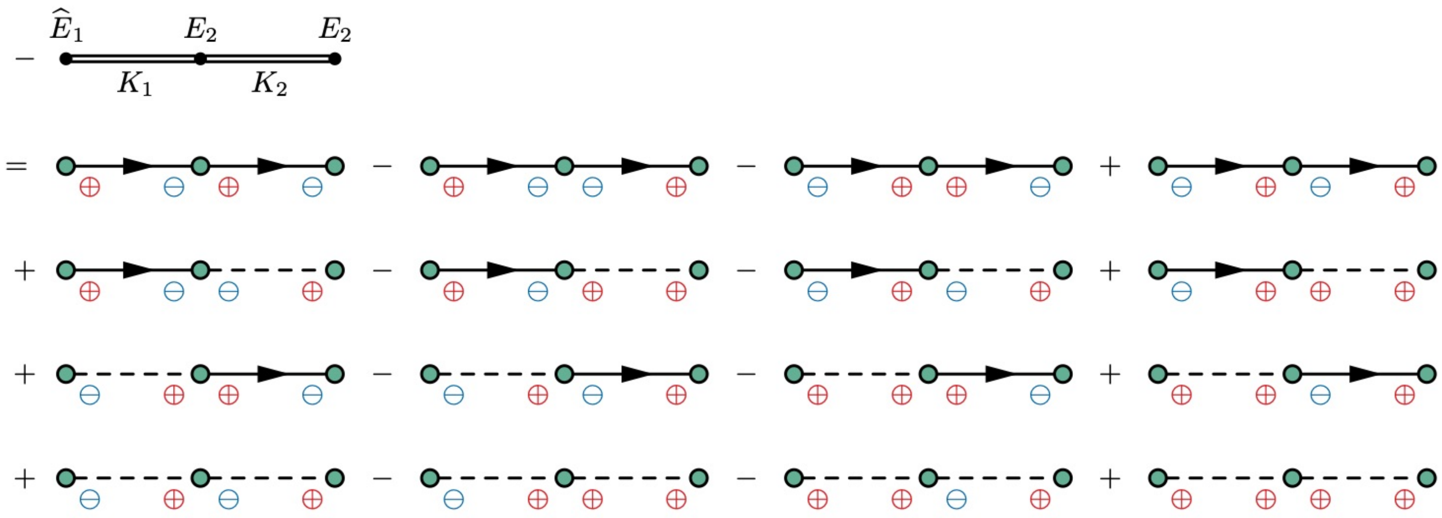
[correlators similar]

1. Fix a partial order
2. Write the uncut tree

[E_K | bar | sign index]

3. Cut! [bar ↔ unbar | remove later index]

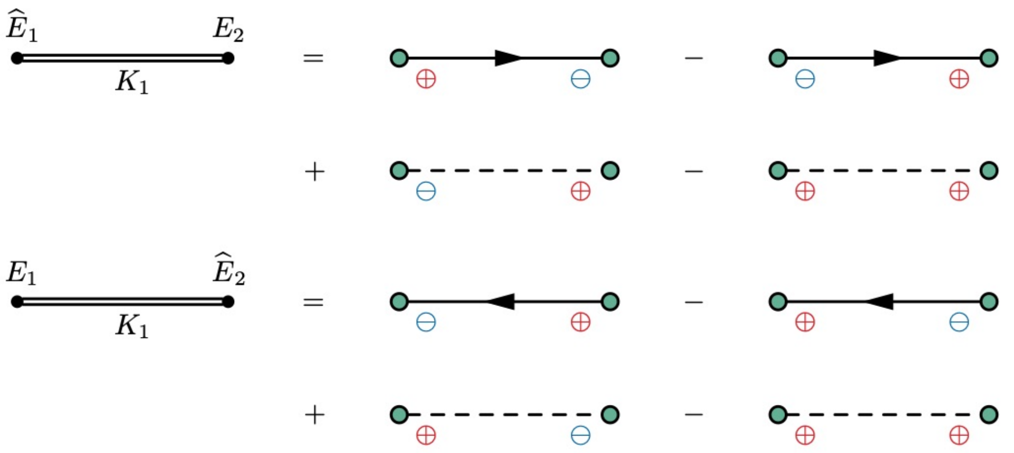
Example: 3-site chain



$$-\widetilde{\psi}_{\text{3-chain}}(\widehat{1}) = \sum_{\mathsf{a},\mathsf{b}=\pm} \mathsf{ab}\Big\{\big[1_{1^\mathsf{a}}2_{\bar{1}^\mathsf{a}2^\mathsf{b}}3_{\bar{2}^\mathsf{b}}\big] + \big[1_{1^\mathsf{a}}2_{\bar{1}^\mathsf{a}\bar{2}^\mathsf{b}}\big]\big[3_2\big] + \big[1_{\bar{1}^\mathsf{a}}\big]\big[2_{12^\mathsf{b}}3_{\bar{2}^\mathsf{b}}\big] + \big[1_{\bar{1}^\mathsf{a}}\big]\big[2_{1\bar{2}^\mathsf{b}}\big]\big[3_2\big]\Big\}$$

$$2_{\bar{1}^\mathsf{a}2^\mathsf{b}} \equiv E_2 - \mathsf{a}K_1 + \mathsf{b}K_2$$

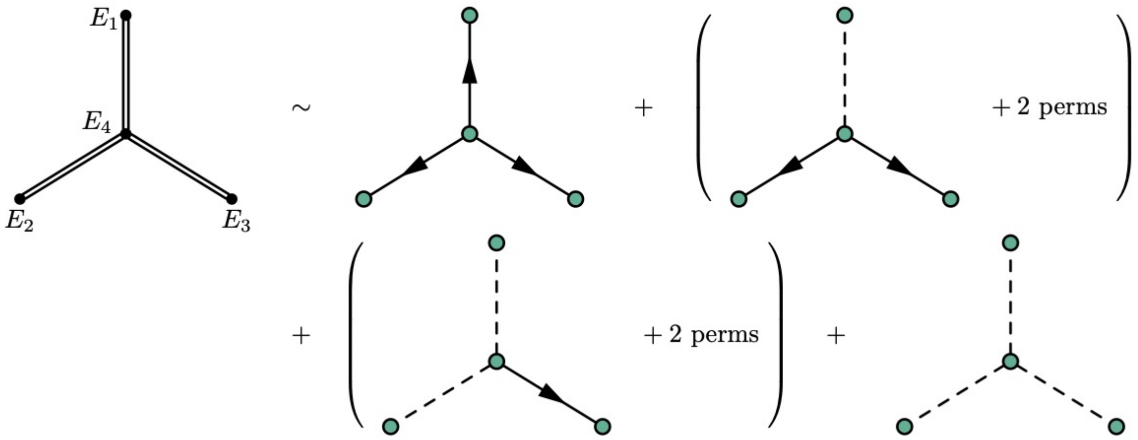
2-site chain:



$$\widetilde{\psi}_2(\widehat{1}) = \sum_{\mathsf{a}=\pm} \mathsf{a}\left\{\left[1_{1^{\mathsf{a}}}2_{\bar{1}^{\mathsf{a}}}\right] + \left[1_{\bar{1}^{\mathsf{a}}}\right]\left[2_1\right]\right\}$$

$$\widetilde{\psi}_2(\widehat{2}) = \sum_{\mathsf{a}=\pm} \mathsf{a}\left\{\left[2_{1^{\mathsf{a}}}1_{\bar{1}^{\mathsf{a}}}\right] + \left[2_{\bar{1}^{\mathsf{a}}}\right]\left[1_1\right]\right\}$$

4-site star:



$$\widetilde{\psi}_{\text{4-star}} = \sum_{\mathsf{a,b,c}=\pm} \mathsf{abc}\Big\{\left[4_{1^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}}(1_{\bar{1}^{\mathsf{a}}})(2_{\bar{2}^{\mathsf{b}}})(3_{\bar{3}^{\mathsf{c}}})\right] + \Big(\left[4_{\bar{1}^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}}(2_{\bar{2}^{\mathsf{b}}})(3_{\bar{3}^{\mathsf{c}}})\right]\left[1\right] + 2\text{ perms}\Big)$$
$$+ \Big(\left[4_{\bar{1}^{\mathsf{a}}\bar{2}^{\mathsf{b}}3^{\mathsf{c}}}3_{\bar{3}^{\mathsf{c}}}\right]\left[1_1\right]\left[2_2\right] + 2\text{ perms}\Big) + \left[4_{\bar{1}^{\mathsf{a}}\bar{2}^{\mathsf{b}}\bar{3}^{\mathsf{c}}}\right]\left[1_1\right]\left[2_2\right]\left[3_3\right]\Big\}$$

family tree vs energy integral

FRW conformal amplitudes => Twisted integrals of flat amplitudes

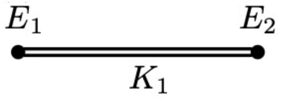
[Arkani-Hamed et al. 2312.05303]

$$\mathcal{T} \sim \int_0^\infty \frac{\mathrm{d}\epsilon_1 \cdots \mathrm{d}\epsilon_V}{(\epsilon_1 \cdots \epsilon_V)^q} \times \{\text{energy integrand}\}$$

The energy integrand constructable recursively [cosmological polytope]

[Arkani-Hamed et al. 1709.02813]

Example: 2-site wavefunction



$$\frac{2K_1}{\mathcal{E}_{12}(\mathcal{E}_1 + K_1)(\mathcal{E}_2 + K_1)} \qquad \mathcal{E}_i \equiv E_i + \epsilon_i$$

---

1. Time and energy integrals essentially related by Fourier transform

2. Familty trees to family chains.

3. Chain diagrams directly reducible:

$$[1 \cdots N] = (-\mathrm{i})^N \int_0^\infty \prod_{i=1}^{N} \left[\frac{\mathrm{d}\epsilon_i \, (\mathrm{i}\epsilon_i)^{-q_i}}{\Gamma[1-q_i]} \right] \frac{1}{\mathcal{E}_1 \mathcal{E}_{12} \cdots \mathcal{E}_{1 \cdots N}} \equiv \{1 \cdots N\}$$

Family tree decomposition + chain fractions {1…N} recover the “polytope recursion”

Inflationary limit

Interesting to consider the special case of ϕ^3 theory in dS limit (all q = 0)

Boundary of IR safe region: A family tree of V sites contains q = 0 poles up to deg V
All poles cancel out in amplitudes, finite terms being polylogs

Example 2-site wavefunction: $\widetilde{\psi}_{\text{2-chain}} = [1_1 2_{\bar{1}}] - [1_{\bar{1}} 2_1] + [1_{\bar{1}}][2_1] - [1_1][2_1]$

$$[1_1 2_{\bar{1}}]_{q_1=q_2=q} = \frac{-1}{[\mathrm{i}(E_1+K)]^{2q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma[n+2q]}{n+q} \left(\frac{E_2-K}{E_1+K}\right)^n$$

$$= \underbrace{-\frac{1}{2q^2} + \frac{\gamma_E + \log[\mathrm{i}(E_1+K)]}{q}}_{\text{divergent terms}} \underbrace{- \mathrm{Li}_2 \frac{K-E_2}{K+E_1} - \Big(\log[\mathrm{i}(E_1+K)] + \gamma_E\Big)^2 - \frac{\pi^2}{6}}_{\text{finite terms}} + \mathcal{O}(q)$$

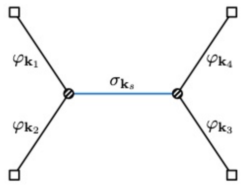
Final answer: $\widetilde{\psi}_{\text{2-chain}} = \mathrm{Li}_2 \frac{E_2-K}{E_{12}} + \mathrm{Li}_2 \frac{E_1-K}{E_{12}} + \log \frac{E_1+K}{E_{12}} \log \frac{E_2+K}{E_{12}} - \frac{\pi^2}{6}$

More sites: integrated polylogs could be tedious, but easy to get the symbol

[See also Hillman 1912.09450]

Concluding remarks

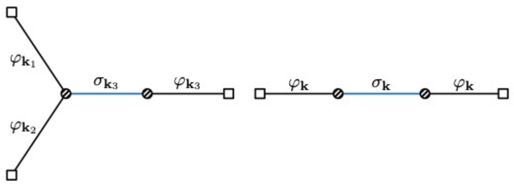
Analytical progress of cosmological correlators from our group since 2022:



boost-less graphs

PMB / bootstrap

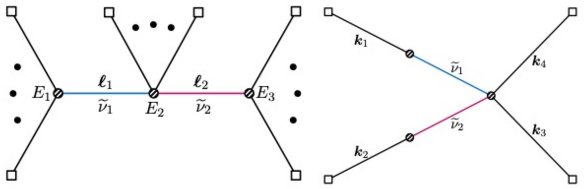
[2205.01692; 2208.13790]



Closed-form formula

improved bootstrap

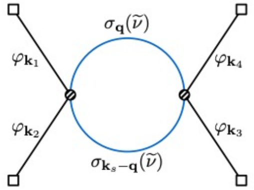
[2301.07047]



Multiple massive exchange

family-tree decomposition

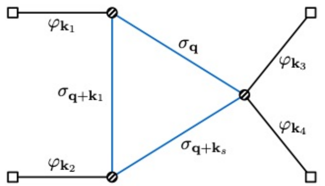
[2309.10849]

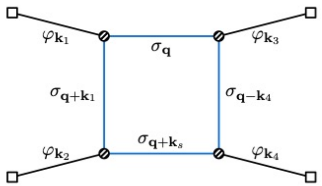


1-loop bubble graphs

spectral decomposition

[2211.03810]

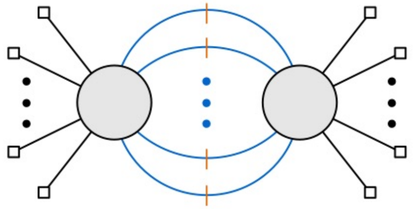




1-loop signal

PMB, bootstrap

[2304.13295]



All-loop signal

factorization theorem

[2308.14802]

Analytical results enable fast numerical implementation:

1-loop: Brute-force numerical [$O(10^5)$ CPU hours] vs. Analytical [O(10s) on a laptop]

[Lian-Tao Wang, **ZX**, Yi-Ming Zhong, 2109.14635] [**ZX**, Hongyu Zhang, 2211.03810]

Yet still a lot more to be understood. Far from done!

Concluding remarks

✓ Arbitrary massive trees essentially solved [PMB + family tree];

✓ Nonlocal signals in arbitrary graphs at all loop orders obtained [PMB + factorization]

✓ A new class of special functions identified (family trees), many new math structures!

✓ FRW conformal amplitudes obtained: many lessons learnt from a good toy model!

Progress underway:

- 1-loop bubble done; other simple loops (relevant to pheno) doable as well
- Nonlocal signals found; local signals? Analyticity of arbitrary graphs? Dispersion!
- Beyond dS: Slow-roll correction / cosmo collider in non-inflation scenarios
- Analytical-result-inspired template design: pinch and cut; phase information

Rich mathematical structure ↔ deep physics of QFT in cosmological background

Thank you!

Back up

1-loop bubble from spectral decomposition

$$\mathcal{L}_{\varphi,\widetilde{\nu}} = \frac{1}{16k_1k_2k_3k_4(k_{12}k_{34})^{5/2}}\Big[\widehat{\mathcal{J}}_{\rm NS}(r_1,r_2)+\widehat{\mathcal{J}}_{\rm LS}(r_1,r_2)+\widehat{\mathcal{J}}_{\rm BG}(r_1,r_2)\Big].$$

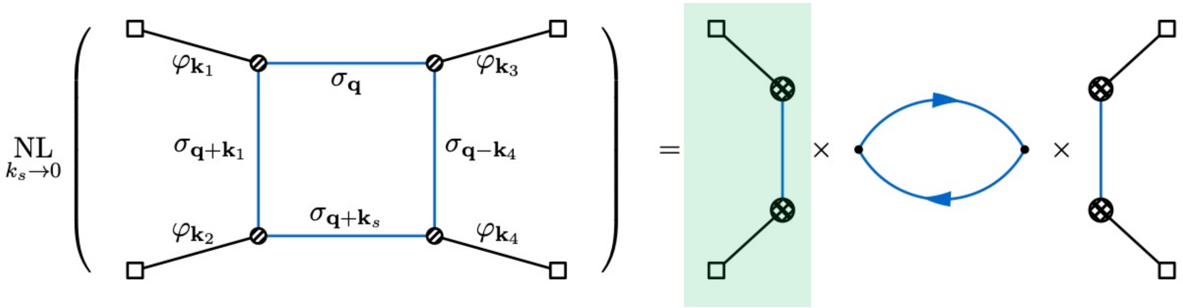
$$\widehat{\mathcal{J}}_{\rm NS} = \frac{2(r_1r_2)^{3/2+2\mathrm{i}\widetilde{\nu}}}{\pi^2\cos(2\pi\mathrm{i}\widetilde{\nu})}\sum_{n=0}^{\infty}\frac{(1+n)_{\frac12}\big[(1+\mathrm{i}\widetilde{\nu}+n)_{\frac12}\big]^2(1+2\mathrm{i}\widetilde{\nu}+n)_{\frac12}}{(1+2\mathrm{i}\widetilde{\nu}+2n)_2}(\tfrac32+2\mathrm{i}\widetilde{\nu}+2n)$$
$$\times\,{}_2\mathcal{F}_1\left[\begin{matrix}2+\mathrm{i}\widetilde{\nu}+n,\frac52+\mathrm{i}\widetilde{\nu}+n\\ \frac52+2\mathrm{i}\widetilde{\nu}+2n\end{matrix}\middle|r_1^2\right]{}_2\mathcal{F}_1\left[\begin{matrix}2+\mathrm{i}\widetilde{\nu}+n,\frac52+\mathrm{i}\widetilde{\nu}+n\\ \frac52+2\mathrm{i}\widetilde{\nu}+2n\end{matrix}\middle|r_2^2\right](r_1r_2)^{2n}+\text{c.c.}.$$

$$\widehat{\mathcal{J}}_{\rm LS} = -\frac{2(r_1/r_2)^{3/2+2\mathrm{i}\widetilde{\nu}}}{\pi^2\cos(2\pi\mathrm{i}\widetilde{\nu})}\sum_{n=0}^{\infty}\frac{(1+n)_{\frac12}\big[(1+\mathrm{i}\widetilde{\nu}+n)_{\frac12}\big]^2(1+2\mathrm{i}\widetilde{\nu}+n)_{\frac12}}{(1+2\mathrm{i}\widetilde{\nu}+2n)_2}(\tfrac32+2\mathrm{i}\widetilde{\nu}+2n)$$
$$\times\,{}_2\mathcal{F}_1\left[\begin{matrix}2+\mathrm{i}\widetilde{\nu}+n,\frac52+\mathrm{i}\widetilde{\nu}+n\\ \frac52+2\mathrm{i}\widetilde{\nu}+2n\end{matrix}\middle|r_1^2\right]{}_2\mathcal{F}_1\left[\begin{matrix}\frac12-\mathrm{i}\widetilde{\nu}-n,1-\mathrm{i}\widetilde{\nu}-n\\ -\frac12-2\mathrm{i}\widetilde{\nu}-2n\end{matrix}\middle|r_2^2\right]\left(\frac{r_1}{r_2}\right)^{2n}+\text{c.c.}.$$

$$\widehat{\mathcal{J}}_{\rm BG} = \sum_{\ell,m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{\ell+n+1}(\ell+1)_{2m+4}(\frac52+\ell+2n)}{2^{2m}n!(m-n)!\big(\frac52+\ell+n\big)_{m+1}}$$
$$\times\left[\widehat{\rho}^{\rm dS}_{\widetilde{\nu}}(-\tfrac{\mathrm{i}5}{2}-\mathrm{i}\ell-2\mathrm{i}n)-\frac{1}{(4\pi)^2}\log\mu_R^2\right]r_1^{2m}\left(\frac{r_1}{r_2}\right)^{5/2+\ell}.$$

1-loop box graph:

$$\Delta\mathscr{L} = \frac{1}{2}a^3\varphi'\sigma^2$$



$$\mathcal{T}_{\mathsf{c}}^{(\mathrm{L})}(k_1,k_2) = \frac{1}{4k_1k_2}\sum_{\mathsf{a}_1,\mathsf{a}_2=\pm}(-\mathsf{a}_1\mathsf{a}_2)\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2}\frac{d\tau_2}{\tau_2^2}e^{\mathsf{a}_1ik_1\tau_1+\mathsf{a}_2ik_2\tau_2}D^{(\widetilde{\nu})}_{\mathsf{a}_1\mathsf{a}_2}(k_1;\tau_1,\tau_2)$$
$$\times(-\tau_1)^{3/2+\mathrm{c}\mathrm{i}\widetilde{\nu}}(-\tau_2)^{3/2+\mathrm{c}\mathrm{i}\widetilde{\nu}}$$

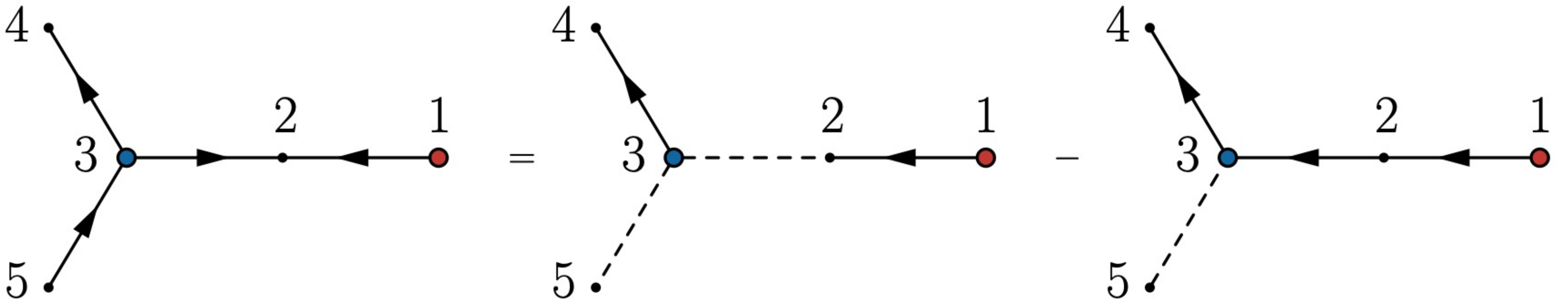
Computed with an improved bootstrap method in 2301.07047

$$\mathcal{T}_{\mathsf{c}}^{(\mathrm{L})}(k_1,k_2) = -\frac{8(2+\mathrm{ic}\widetilde{\nu})\Gamma(2+2\mathrm{ic}\widetilde{\nu})\sin(\pi\mathrm{ic}\widetilde{\nu})}{3+2\mathrm{ic}\widetilde{\nu}}\frac{k_{12}^{-4-2\mathrm{c}\mathrm{i}\widetilde{\nu}}}{4k_1k_2}$$

The right subgraph is similar. Putting everything together, we get:

$$\lim_{k_s\to 0}\Big[\mathcal{T}_{\mathrm{box}}(\{\mathbf{k}\})\Big]_{\mathrm{NL}} = -\frac{k_s^3}{2(4\pi)^{7/2}k_1k_2k_3k_4k_{12}^4k_{34}^4}\left(\frac{k_s^2}{4k_{12}k_{34}}\right)^{2\mathrm{i}\widetilde{\nu}}\frac{(2+\mathrm{i}\widetilde{\nu})^4}{(3+2\mathrm{i}\widetilde{\nu})^2}\sinh^2(\pi\widetilde{\nu})$$
$$\times\Gamma\left[3+2\mathrm{i}\widetilde{\nu},-\frac{3}{2}-2\mathrm{i}\widetilde{\nu}\right]\Gamma^2\left[\frac{3}{2}+\mathrm{i}\widetilde{\nu},-2-\mathrm{i}\widetilde{\nu}\right]+\text{c.c.}.$$

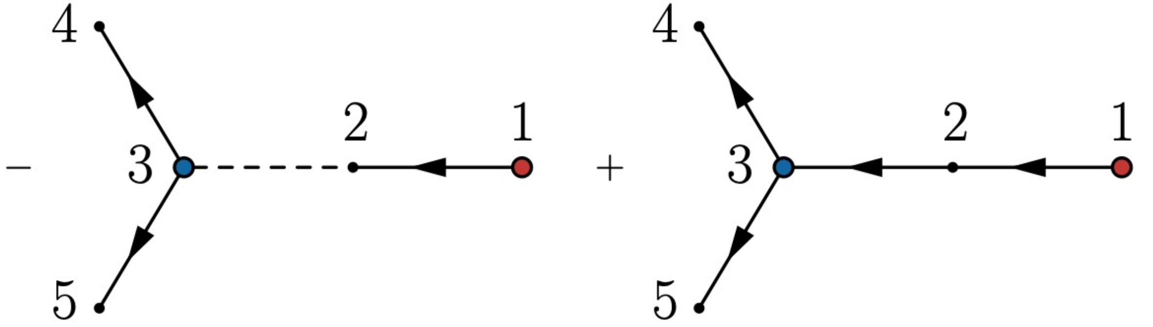
Example: a 5-fold int



Choose Site 1 as the earliest

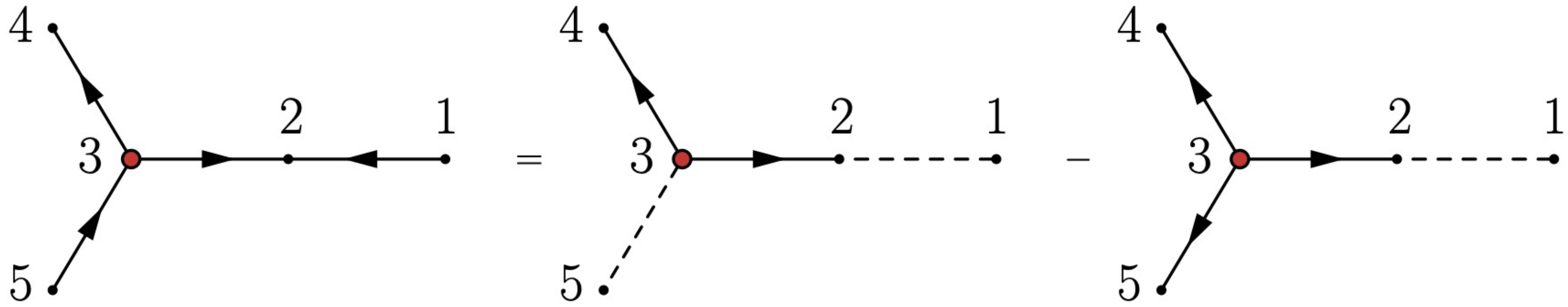
1->2 good 2->3 flip

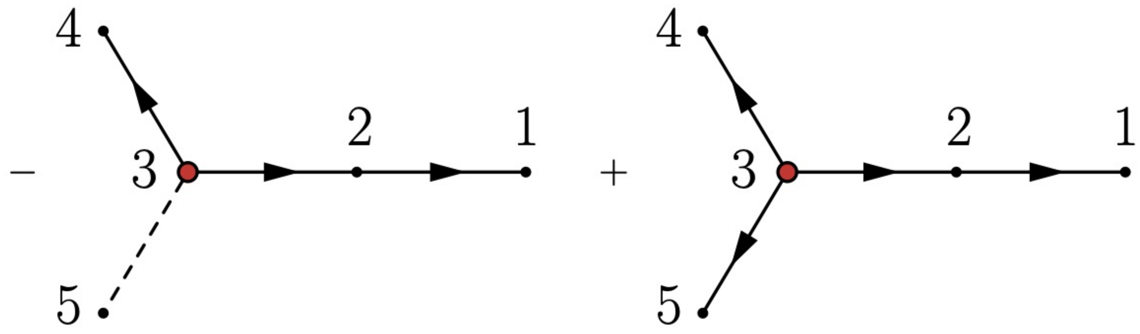
3->4 good 4->5 flip



[Also need to decide “locally” earliest site in all nested subgraph, in this case Site 3]

Example: a 5-fold int





We can as well choose a differerent site as the earliest

Say Site 3:

3->4 good 3->2 good

3->5 flip 2->1 flip

For a tree graph: choosing an earliest site fixes the partial order

Why is the rule correct?

Starting from the original Feynman rule. The bulk propagator reads:

$$\widetilde{G}(K_\ell;\tau_A,\tau_B)=e^{-\mathrm{i}K_\ell(\tau_A-\tau_B)}\theta(\tau_A-\tau_B)+e^{+\mathrm{i}K_\ell(\tau_A-\tau_B)}\theta(\tau_B-\tau_A)-e^{+\mathrm{i}K_\ell(\tau_A+\tau_B)}$$

When a partial order P given, we adjust G to make it consistent with P

Say, when A earlier than B:

$$\widetilde{G}(K_\ell;\tau_A,\tau_B)=\left[e^{+\mathrm{i}K_\ell(\tau_A-\tau_B)}-e^{-\mathrm{i}K_\ell(\tau_A-\tau_B)}\right]\theta(\tau_B-\tau_A)+e^{-\mathrm{i}K_\ell(\tau_A-\tau_B)}-e^{+\mathrm{i}K_\ell(\tau_A+\tau_B)}$$

Four terms, packed into:

$$\widetilde{G}(K_\ell;\tau_A,\tau_B)=\sum_{\mathsf{a}=\pm}\mathsf{a}\left[e^{\mathrm{i}\mathsf{a}K_\ell(\tau_A-\tau_B)}\theta(\tau_B-\tau_A)+e^{-\mathrm{i}\mathsf{a}K_\ell\tau_A+\mathrm{i}K_\ell\tau_B}\right]$$ → uncut + cut

In our family tree notations: $\sum_{\mathsf{a}=\pm}\mathsf{a}\Big\{[\cdots A_{\ell^{\mathsf{a}}}B_{\bar{\ell}^{\mathsf{a}}}\cdots]+[\cdots A_{\bar{\ell}^{\mathsf{a}}}][B_\ell\cdots]\Big\}$

Every tree graph with V vertex has 4^{V-1} terms [consistent w/ results from diff eq (2312.05303)]

More sites: integrated polylogs could be tedious, but easy to get the symbol

A 3-site example: $[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}} \sum_{n_1,n_3=0}^{\infty} \frac{(-1)^{n_{13}}\Gamma[n_{13}+q_{123}]}{(n_1+q_1)(n_3+q_3)} \frac{u_1^{n_1}}{n_1!} \frac{u_3^{n_3}}{n_3!} \qquad u_i \equiv \omega_i/\omega_2$

Inflationary limit: $\lim_{q\to 0} [2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{3q}} \left\{ \frac{\Gamma[3q]}{q^2} + \frac{\mathrm{Li}_2(-u_1)+\mathrm{Li}_2(-u_3)}{q} + \mathbf{L}_3(u_1,u_3) \right\} + \mathcal{O}(q)$

$$\mathbf{L}_3(u_1,u_3) \equiv \sum_{n_1,n_3=1}^{\infty} \frac{(-1)^{n_{13}}\Gamma[n_{13}]}{n_1 n_3} \frac{u_1^{n_1}}{n_1!} \frac{u_3^{n_3}}{n_3!}$$

$\mathbf{L}_3$ is a weight-3 polylog: $\frac{\partial}{\partial \log u_1} \frac{\partial}{\partial \log u_3} \mathbf{L}_3(u_1,u_3) = \log(1+u_1) + \log(1+u_3) - \log(1+u_{13})$

$$\mathcal{S}(\mathbf{L}_3) = \frac{(1+u_1)(1+u_3)}{1+u_{13}} \otimes u_1 \otimes u_3 + \frac{(1+u_1)(1+u_3)}{1+u_{13}} \otimes u_3 \otimes u_1 + \frac{1+u_{13}}{1+u_1} \otimes (1+u_1) \otimes u_1 + \frac{1+u_{13}}{1+u_3} \otimes (1+u_3) \otimes u_3$$

Question: a new algorithm for all family-trees/amplitudes? [See also Hillman 1912.09450]