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Modifying gravity without extra degrees of freedom

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Quantum Gravity and Cosmology 2024

ShanghaiTech University, Shanghai

July 2, 2024

[Based on: 1910.13995, 2011.00805, 2104.07615, 2302.02090]

Why modifying gravity?

Phenomenological side:

- (2011 Nobel prize) To explain the early and late accelerated expansion of our universe.
- (2017 Nobel prize) The gravitational waves have been detected, which are new tools to test gravity theories.

Theoretical side:

To check the conditions that GR is based on, so that we can understand if and why GR is the unique theory of gravity.

Modified gravity: playing with DoF’s

The key question:

Keeping the correct degrees of freedom (DoF’s).

To introduce the wanted DoF’s.

To eliminate the unwanted (ghost-like) DoF’s.

Two faces of modified gravity

(2011 Nobel prize)
Dark energy / Inflation

Extra mode(s) without ghost(s).

⬇

"Gravity" is more than GR.

(Horndeski, DHOST, Horava, EFT of inflation, dRGT, SCG, U-DHOST ...)

Example: extra scalar mode without ghost

1915 — GR

$$\mathcal{L} = \frac{1}{16\pi G} R$$

1961 — Brans-Dicke

[Brans & Dicke, 1961]

$$\mathcal{L} = \phi R - \frac{\omega}{\phi} (\partial\phi)^2$$

1999 — *k*-essence

[Chiba, Okabe, Yamaguchi, 1999]

[Armendariz-Picon, Damour, Mukhanov, 1999]

$$\mathcal{L} = g(\phi)\, R + F(\phi, \partial\phi)$$

2011 — Horndeski

*[Deffayet, **XG**, Steer & Zahariade **Phys.Rev. D** 84 (2011) 064039]*

2015 — DHOST

[Langlois & Noui, 2015]

[Crisostomi, Koyama & Tasinato, 2016]

$$\begin{aligned}\mathcal{L} = {} & G_2(X,\phi) + G_3(X,\phi)\Box\phi \\ & + G_4(X,\phi)\, R + \frac{\partial G_4}{\partial X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ & + G_5(X,\phi)\, G^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\ & - \frac{1}{6}\frac{\partial G_5}{\partial X}\left[(\Box\phi)^3 - 3\Box\phi\,(\nabla_\mu\nabla_\nu\phi)^2 + 2\,(\nabla_\mu\nabla_\nu\phi)^3\right]\end{aligned}$$

- Higher derivatives in Lagrangians/EoMs,
- Propagating 1 scalar + 2 tensor DoFs.

Two faces of modified gravity

(2011 Nobel prize)
Dark energy / Inflation

Extra mode(s) without ghost(s).

↓

"Gravity" is more than GR.

(Horndeski, DHOST, Horava, EFT of inflation, dRGT, SCG, U-DHOST …)

(2017 Nobel prize)
Gravitational waves (GWs)

Non-GR theory for the two tensorial degrees of freedom (TTDOFs).

↓

"Gravity" is just the TTDOFs, but behave differently from that of GR.

(Cuscuton, MMG, 4dEGB, TTDOF, …)

Uniqueness of GR

Einstein equation is quite unique.

$$\alpha G_{\mu\nu} + \lambda g_{\mu\nu} = 0$$

[Lovelock, 1971]

Any metric theory of gravity alternative to GR must satisfy (at least):

- extra degrees of freedom,
- extra dimensions (e.g., brane world),
- higher derivative terms (e.g., $f(R)$),
- non-Riemannian geometry (e.g., $f(T)$),
- giving up locality.

Uniqueness of GR

GR is the unique theory (kinetic term) for the TTDOFs, if we require Lorentz invariance and locality.

Degrees of freedom in GR:

10 - 4 - 4 = 2

4 gauge modes

4 constraints

2 polarizations of a massless graviton

T**T**DOFs = **T**ransverse and **T**raceless DOFs
= **T**wo **T**ensorial DOFs

Uniqueness of GR

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

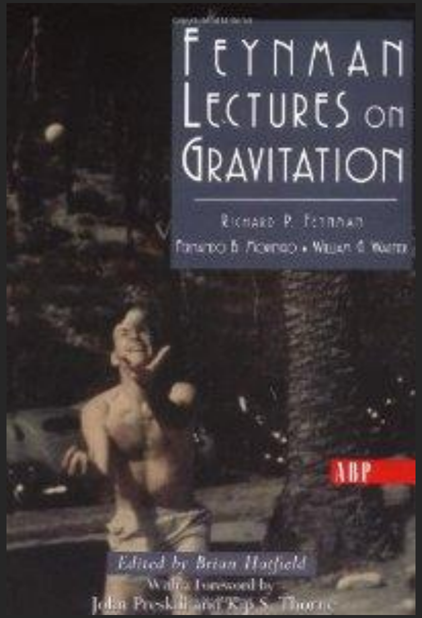
$$\mathcal{L}_2 = c_1\, \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + c_2\, \partial_\mu h^{\mu\nu} \partial_\nu h + c_3\, \partial_\nu h^{\mu\nu} \partial^\lambda h_{\mu\lambda} + c_4\, \partial_\mu h \partial^\mu h$$

$$\delta_{\boldsymbol{\xi}} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$c_1 = -\frac{1}{4}, \qquad c_2 = -\frac{1}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4}.$$

"resummation" → GR

$$\mathcal{L} = \sum_n \mathcal{L}_n = \sqrt{-\det\left(\eta_{\mu\nu} + h_{\mu\nu}\right)} R\left[\eta_{\mu\nu} + h_{\mu\nu}\right] \equiv \sqrt{g} R[g]$$



1962-1963

Uniqueness of GR

How about massive gravitons (5 DOFs)?

For the mass (potential) terms:

$$\mathcal{L}_2^{(\mathrm{p})} = -\frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) \equiv \mathcal{L}_{\mathrm{FP}}$$

(Fierz-Pauli mass term)

$$\sum_n \mathcal{L}_n^{(\mathrm{p})} \simeq \sqrt{-g}\mathcal{L}_{\mathrm{dRGT}}$$

[C. de Rham, G. Gabadadze, A. Tolley, 1011.1232]

For the kinetic term:

Very likely GR is also the unique kinetic term for both massless and massive gravitons, with Lorentz invariance and locality.

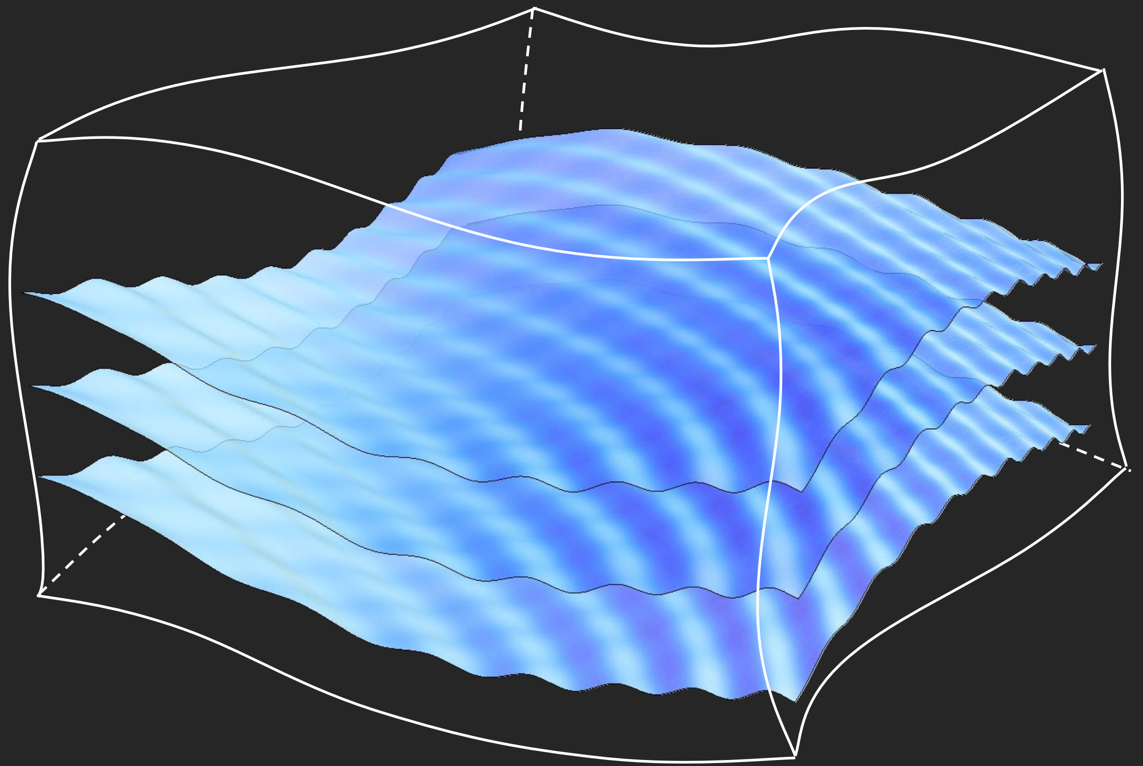
[C. de Rham, A. Matas, A. Tolley, 1311.6485]

*[**XG**, 1403.6781]*

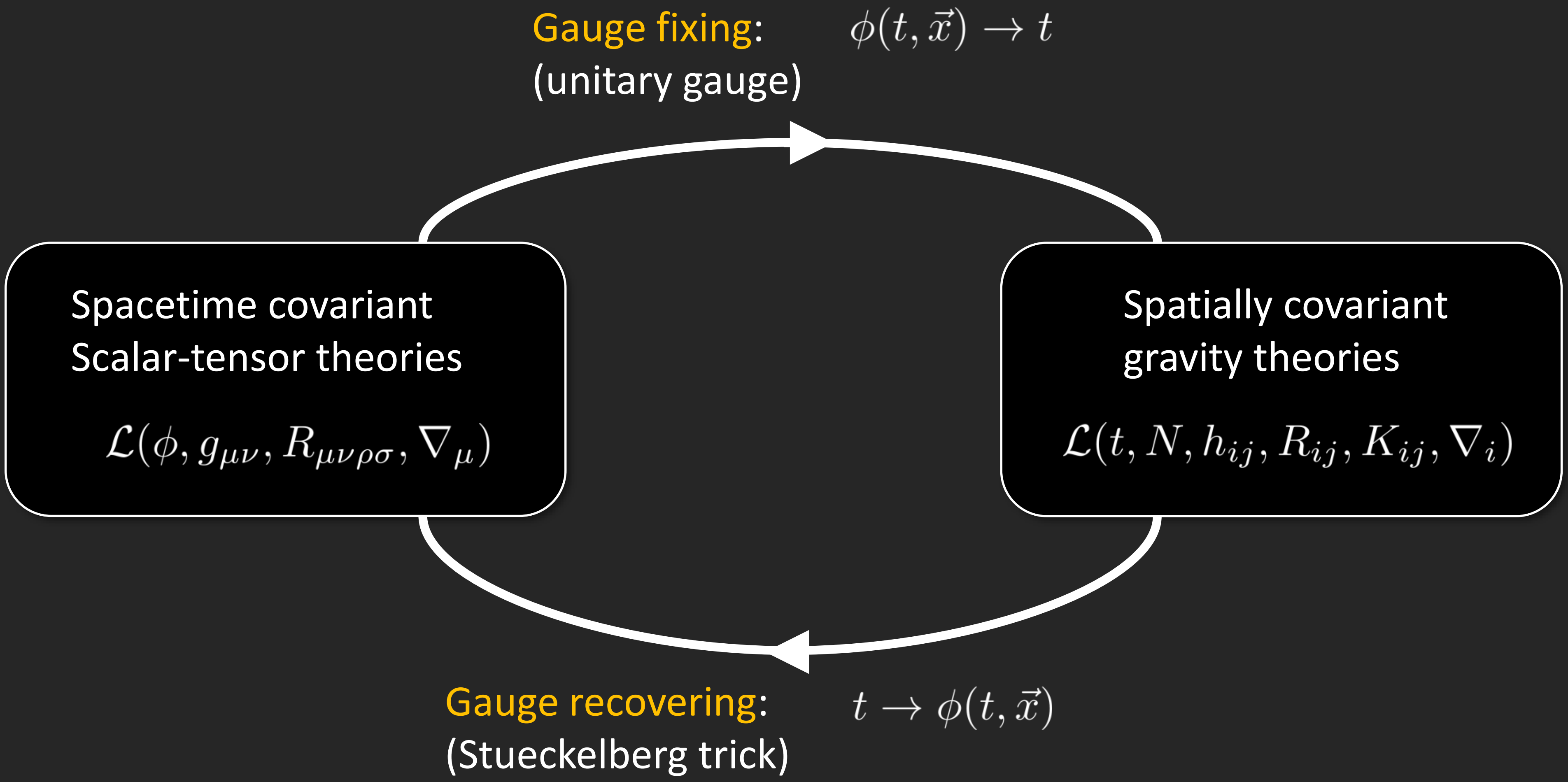
Scalar-tensor theory v.s. Spatially covariant gravity

How about to abandon Lorentz invariance?

With only spatial covariance?



Scalar-tensor theory v.s. Spatially covariant gravity



[H. Motohashi, T. Suyama, K. Takahashi, 2016]
[A. De Felice, D. Langlois, S. Mukohyama, K. Noui & A. Wang, 2018]

Spatially covariant gravity

2004 — **Ghost condensation**
[Arkani-Hamed, Cheng, Luty & Mukohyama]

2007 — **Effective field theory of inflation**
[Cheung, Creminelli, Fitzpatrick, Kaplan & Senatore]

2009 — **Hořava gravity**
[P. Horava]

2014 — **GLPV theory**
*[**G**leyzes, **L**anglois, **P**iazza & **V**ernizzi]*

2014 — **Spatially covariant gravity**
[XG, 1406.0822]

Spatially covariant gravity

[XG, et al: 1406.0822 , 1409.6708, 1806.02811, 1902.07702, 1910.13995, 2004.07752, 2006.15633, 2111.08652 …]

Spatially covariant gravity

$$S=\int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\mathcal{L}(t,N,h_{ij},R_{ij},K_{ij},\nabla_i)$$

2 tensor + 1 scalar DoFs with higher derivative EoMs,

Spatially covariant gravity

[XG, et al: 1406.0822 , 1409.6708, 1806.02811, 1902.07702, 1910.13995, 2004.07752, 2006.15633, 2111.08652 …]

Spatially covariant gravity

$$S = \int \mathrm{d}t\mathrm{d}^3x N\sqrt{h}\mathcal{L}(t, N, h_{ij}, R_{ij}, K_{ij}, \nabla_i)$$

2 tensor + 1 scalar DoFs with higher derivative EoMs,
as long as the Lagrangian is nonlinear in the lapse function N.

Spatially covariant gravity

Question:

Does it exist a theory of spatially covariant gravity,
without the extra scalar mode?

Cuscuton as a special case of SCG

2007 • Cuscuton

[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}R + \mu^2\sqrt{|-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi|} - V(\phi)\right)$$

Only 2 DOFs (no scalar) when the scalar field is timelike (e.g., in a cosmological background).

[H. Gomes, D. Guariento, 1703.08226]

The scalar mode becomes an instantaneous mode (with an infinite speed of sound) and effectively non-dynamical.

[A. De Felice, D. Langlois, S. Mukohyama, K. Noui, A. Wang, 1803.06241]

$$L = \frac{1}{2}\left(\dot{\psi}^2 - c_s^2\left(\partial_i\psi\right)^2\right) \qquad \ddot{\psi} + c_s^2\partial^2\psi^2 = 0 \quad \xrightarrow{c_s \to \infty} \quad \partial^2\psi^2 = 0$$

Cuscuton as a special case of SCG

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[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

$$S=\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}R+\mu^2\sqrt{|-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi|}-V(\phi)\right)$$

$\phi = t$ (unitary gauge)

$$S^{\mathrm{u.g.}}=\int \mathrm{d}t\mathrm{d}^3x\sqrt{h}\left[\frac{1}{2}N\left(K_{ij}K^{ij}-K+{}^{(3)}R\right)+\mu^2-NV(t)\right]$$

A special case of spatially covariant gravity (linear in lapse function N).

Non-GR theories for the TTDOFs

2007 — Cuscuton

[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

2017 — Minimally modified gravity

[C. Lin, S. Mukohyama, 1708.03757]

$$S = \int \mathrm{d}t\mathrm{d}^3x\sqrt{h}N\,F\left(K_{ij}, R_{ij}, h_{ij}, \nabla_i; t\right)$$

(with additional conditions on F)

Non-GR theories for the TTDOFs

2007 — Cuscuton

[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

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2018 — Extended Cuscuton

[A. Iyonaga, K. Takahashi, T. Kobayashi, 1809.10935]

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2019 Spatially covariant gravity with TTDOFs
*[**XG**, Z.-B. Yao, 1910.13995]*
Perturbative approach *[Yu-Min Hu, **XG**, 2104.07615]*
With the dynamical lapse *[J. Lin, Y. Gong, Y. Lu, F. Zhang, 2011.05739]*

TTDOF conditions in the Lagrange approach

*[**XG**, Z.-B. Yao, 1910.13995]*

For a general SCG action, we got 2 TTDOF conditions.

$$\tilde{S} = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\mathcal{L}\left(N, h_{ij}, K_{ij}, R_{ij}, \nabla_i; t\right)$$

(1) degeneracy condition:

The $\{N, K_{ij}\}$-sector must be degenerate.

$$0 \approx \mathcal{S}\left(\vec{x}, \vec{y}\right) \equiv \frac{\delta^2 S_B}{\delta N\left(\vec{x}\right)\delta N\left(\vec{y}\right)} - \int \mathrm{d}^3x' \int \mathrm{d}^3y' N\left(\vec{x}'\right) \frac{\delta}{\delta N\left(\vec{x}\right)}\left(\frac{1}{N\left(\vec{x}'\right)}\frac{\delta S_B}{\delta B_{i'j'}\left(\vec{x}'\right)}\right) \times \mathcal{G}_{i'j',k'l'}\left(\vec{x}', \vec{y}'\right) N\left(\vec{y}'\right) \frac{\delta}{\delta N\left(\vec{y}\right)}\left(\frac{1}{N\left(\vec{y}'\right)}\frac{\delta S_B}{\delta B_{k'l'}\left(\vec{y}'\right)}\right)$$

(2) consistency condition:

To completely eliminate the single (extra scalar) DOF.

$$0 \approx \mathcal{J}\left(\vec{x}, \vec{y}\right) \equiv \int \mathrm{d}^3x' \int \mathrm{d}^3y' \int \mathrm{d}^3x'' \int \mathrm{d}^3y'' \frac{\delta C\left(\vec{x}\right)}{\delta B_{ij}\left(\vec{x}'\right)} \mathcal{G}_{ij,i'j'}\left(\vec{x}', \vec{x}''\right) \times N\left(\vec{x}''\right) \frac{\delta^2 S_B}{\delta h_{i'j'}\left(\vec{x}''\right)\delta B_{k'l'}\left(\vec{y}''\right)} \mathcal{G}_{k'l',kl}\left(\vec{y}'', \vec{y}'\right) \frac{\delta C\left(\vec{y}\right)}{\delta B_{kl}\left(\vec{y}'\right)} - \int \mathrm{d}^3x' \int \mathrm{d}^3y' \frac{\delta C\left(\vec{x}\right)}{\delta B_{ij}\left(\vec{x}'\right)} \mathcal{G}_{ij,kl}\left(\vec{x}', \vec{y}'\right) N\left(\vec{y}'\right) \frac{\delta C\left(\vec{y}\right)}{\delta h_{kl}\left(\vec{y}'\right)} - \left(\vec{x} \leftrightarrow \vec{y}\right)$$

A concrete example

*[**XG**, Z.-B. Yao, 1910.13995]*

A model quadratic in K_{ij}:

$$S^{(\mathrm{quad})} = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\left[\frac{N}{\beta_2+N}K^{ij}K_{ij} - \frac{1}{3}\left(\frac{2N}{\beta_1+N}+\frac{N}{\beta_2+N}\right)K^2 + \rho_1 + \rho_2 R + \frac{1}{N}\left(\rho_3+\rho_4 R\right)\right],$$

GR: $\beta_1 = \beta_2 = \rho_3 = \rho_4 = 0,\qquad \rho_1 = \mathrm{const.}\qquad \rho_2 = 1$

Cuscuton: $\beta_1 = \beta_2 = \rho_4 = 0,\qquad \rho_2 = 1$

Non-GR theories for the TTDOFs

2007 Cuscuton
[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

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2020 TTDOFs with an auxiliary constraint
*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

2023 TTDOFs with multiple auxiliary constraints
*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

Working directly with the Hamiltonian

(local) Lagrangian

↓

Hamiltonian

↓

TTDOF conditions
(complicated)

(local) Hamiltonian

↓

TTDOF conditions
(simplified)

↓

Lagrangian

A single auxiliary constraint

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

A 20-dim phase space: $\Phi_I = \{N, N^i, h_{ij}\}$ $\quad \Pi^I = \{\pi, \pi_i, \pi^{ij}\}$

$$H_\mathrm{T} = \int \mathrm{d}^3x\sqrt{h}\,\Big[\underbrace{\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla_i\right)}_{\text{free function}} + \underbrace{\nu\varphi\left(N, h_{ij}, \pi^{ij}; \nabla_i\right)}_{\text{auxiliary constraint}} + \underbrace{\lambda\pi}_{N\text{ is non-dyn.}} + \underbrace{N^i\mathcal{H}_i + \lambda^i\pi_i}_{\text{spatial diff.}}\Big]$$

8 constraints: $\pi \approx 0$, $\quad \underbrace{\pi_i \approx 0, \quad \mathcal{H}_i \approx 0,}_{\text{1st class (spatial diff.)}}$ $\quad \varphi \approx 0$

$$\mathcal{H}_i = \sum_I \Pi^I \pounds_{\vec{N}} \Phi_I$$

	π	π_i	$\mathcal{H}_i$	φ
π	X	0	0	X
π_i	0	0	0	0
$\mathcal{H}_i$	0	0	0	0
φ	X	0	0	X

2 second class

6 first class (spatial diff.)

A single auxiliary constraint

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

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$$H_{\mathrm{T}} = \int \mathrm{d}^3x\sqrt{h}\,\Big[\underbrace{\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla_i\right)}_{\text{free function}} + \underbrace{\nu\varphi\left(N, h_{ij}, \pi^{ij}; \nabla_i\right)}_{\text{auxiliary constraint}} + \underbrace{\lambda\pi}_{N\text{ is non-dyn.}} + \underbrace{N^i\mathcal{H}_i + \lambda^i\pi_i}_{\text{spatial diff.}}\Big]$$

8 constraints: $\pi \approx 0,\quad \underbrace{\pi_i \approx 0,\quad \mathcal{H}_i \approx 0,}_{\text{1st class (spatial diff.)}}\quad \varphi \approx 0$

$$\mathcal{H}_i = \sum_I \Pi^I \pounds_{\vec{N}}\Phi_I$$

$$\begin{aligned}\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2\times\#_{\mathrm{var}} - 2\times\#_1 - \#_2\right)\\ &= \frac{1}{2}\left(2\times 10 - 2\times 6 - 2\right)\\ &= 3\end{aligned}$$

Further requirement on φ and/or $\mathcal{H}$.

Case 1

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

If $\pi \approx 0$ is pushed to be a first-class constraint.

We need to require (constraint on both φ and $\mathcal{H}$)

$$\frac{\delta\varphi\left(\vec{y}\right)}{\delta N\left(\vec{x}\right)} \approx 0, \qquad \frac{\delta^2 \int \mathrm{d}^3 z\, \mathcal{H}\left(\vec{z}\right)}{\delta N\left(\vec{x}\right)\delta N\left(\vec{y}\right)} \approx 0$$

The general solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \Big[\mathcal{V}\left(h_{ij}, \pi^{ij}; \nabla\right) + N\mathcal{H}_0\left(h_{ij}, \pi^{ij}; \nabla\right) + \nu\varphi_0\left(h_{ij}, \pi^{ij}; \nabla\right) + \lambda\pi + N^i\mathcal{H}_i + \lambda^i\pi_i\Big]$$

	π	π_i	$\mathcal{H}_i$	φ
π	X	0	0	X
π_i	0	0	0	0
$\mathcal{H}_i$	0	0	0	0
φ	X	0	0	X

Case 1

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The general solution:

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	π	π_i	$\mathcal{H}_i$	φ	$\dot{\pi}$
π	0	0	0	0	0
π_i	0	0	0	0	0
$\mathcal{H}_i$	0	0	0	0	0
φ	0	0	0	X	X
$\dot{\pi}$	0	0	0	X	X

7 first class (π, π_i, $\mathcal{H}_i$)

2 second class (φ, $\dot{\pi}$)

One additional constraint ($\dot{\pi}$)

Case 1

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

If $\pi \approx 0$ is pushed to be a first-class constraint.

We need to require (constraint on both φ and $\mathcal{H}$)

$$\frac{\delta\varphi\left(\vec{y}\right)}{\delta N\left(\vec{x}\right)}\approx 0,\qquad \frac{\delta^{2}\int\mathrm{d}^{3}z\,\mathcal{H}\left(\vec{z}\right)}{\delta N\left(\vec{x}\right)\delta N\left(\vec{y}\right)}\approx 0$$

The general solution:

$$\begin{aligned} H_{\mathrm{T}} &= \int\mathrm{d}^{3}x\Big[\mathcal{V}\left(h_{ij},\pi^{ij};\nabla\right)+N\mathcal{H}_{0}\left(h_{ij},\pi^{ij};\nabla\right)+\nu\varphi_{0}\left(h_{ij},\pi^{ij};\nabla\right)\\ &\quad +\lambda\pi+N^{i}\mathcal{H}_{i}+\lambda^{i}\pi_{i}\Big]\end{aligned}$$

$$\begin{aligned}\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2\times\#_{\mathrm{var}}-2\times\#_{1}-\#_{2}\right)\\ &= \frac{1}{2}\left(2\times 10-2\times(6+1)-(2-1+1)\right)\\ &= 2\end{aligned}$$

Case 2

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

If $\pi \approx 0$ remains to be a second-class constraint.

We need to require (constraint on φ only)

$$\frac{\delta\varphi(\vec{y})}{\delta N(\vec{x})} \approx 0, \qquad \int \mathrm{d}^3 z \left(\frac{\delta\varphi(\vec{x})}{\delta h_{ij}(\vec{z})} \frac{\delta\varphi(\vec{y})}{\delta\pi^{ij}(\vec{z})} - (\vec{x} \leftrightarrow \vec{y}) \right) \approx 0$$

A special solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \left[\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla\right) + \nu\tilde{\varphi}\left(h_{ij}, \pi^{ij}\right) + \lambda\pi + N^i\mathcal{H}_i + \lambda^i\pi_i \right]$$

	π	π_i	$\mathcal{H}_i$	φ
π	X	0	0	X
π_i	0	0	0	0
$\mathcal{H}_i$	0	0	0	0
φ	X	0	0	X

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	π	π_i	$\mathcal{H}_i$	φ	$\dot{\pi}$	$\dot{\varphi}$
π	X	0	0	X	X	X
π_i	0	0	0	0	0	0
$\mathcal{H}_i$	0	0	0	0	0	0
φ	X	0	0	X	X	X
$\dot{\pi}$	X	0	0	X	X	X
$\dot{\varphi}$	X	0	0	X	X	X

6 first class

4 second class

Two additional constraints

Case 2

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

If $\pi \approx 0$ remains to be a second-class constraint.

We need to require (constraint on φ only)

$$\frac{\delta\varphi(\vec{y})}{\delta N(\vec{x})} \approx 0, \qquad \int \mathrm{d}^3 z \left(\frac{\delta\varphi(\vec{x})}{\delta h_{ij}(\vec{z})} \frac{\delta\varphi(\vec{y})}{\delta\pi^{ij}(\vec{z})} - (\vec{x} \leftrightarrow \vec{y}) \right) \approx 0$$

A special solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \left[\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla\right) + \nu\tilde{\varphi}\left(h_{ij}, \pi^{ij}\right) + \lambda\pi + N^i\mathcal{H}_i + \lambda^i\pi_i \right]$$

$$\begin{aligned} \#_{\mathrm{DOF}} &= \frac{1}{2}\left(2 \times \#_{\mathrm{var}} - 2 \times \#_1 - \#_2\right) \\ &= \frac{1}{2}\left(2 \times 10 - 2 \times 6 - (2+2)\right) \\ &= 2 \end{aligned}$$

Multiple auxiliary constraints

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]

A general Hamiltonian with multiple auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big[\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right) + N^i \mathcal{H}_i + \lambda^i \pi_i + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij} \Big]$$

spatial covariance: $N^i \mathcal{H}_i + \lambda^i \pi_i$

Auxiliary constraints: $+\mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij}$

(scalar)	$\mathcal{S}^{\mathrm{n}} \approx 0^{\mathrm{n}},$	$\mathrm{n} = 1, \cdots, \mathcal{N}$
(vector)	$\mathcal{V}^{\mathrm{m}}_i \approx 0^{\mathrm{m}}_i,$	$\mathrm{m} = 1, \cdots, \mathcal{M}$
(tensor)	$\mathcal{T}^{\mathrm{r}}_{ij} \approx 0^{\mathrm{r}}_{ij},$	$\mathrm{r} = 1, \cdots, \mathcal{R}$

We assume them to be second class (the general case).

Multiple auxiliary constraints

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

$$\#_{\mathrm{dof}} = (2_{\mathrm{t}} - \mathcal{R}_{\mathrm{t}}) - (\mathcal{M}_{\mathrm{v}} + \mathcal{R}_{\mathrm{v}}) + \frac{1}{2}(4_{\mathrm{s}} - \mathcal{N}_{\mathrm{s}} - \mathcal{M}_{\mathrm{s}} - 2\times\mathcal{R}_{\mathrm{s}})$$

tensor vector scalar

If no auxiliary constraints: 4 dof = 2t+2s

We need to require: $2-\mathcal{R}\geq 0, \quad \mathcal{M}+\mathcal{R}\leq 0$ $\quad 4-\mathcal{N}-\mathcal{M}-2\mathcal{R}\geq 0$

$$\boxed{\mathcal{R}=0, \quad \mathcal{M}=0, \quad \mathcal{N}\leq 4}$$

No vector nor tensor constraints;
No more than 4 scalar constraints.

Multiple auxiliary constraints

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]

A general Hamiltonian with scalar auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \left(\mathscr{H} + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + N^i \mathcal{H}_i + \lambda^i \pi_i \right)$$

$$\#_{\mathrm{dof}} = 2_{\mathrm{t}} + \frac{1}{2} \left(4_{\mathrm{s}} - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}} \right)$$

$$4 - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}} = 0$$

$$\mathcal{N} \leq \#^{\mathrm{s}}_{\mathrm{1st}} + \#^{\mathrm{s}}_{\mathrm{2nd}} \leq 4$$

Classification of TTDOF theories

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

The "minimalizing" and "symmetrizing" conditions.

TABLE I. The minimalizing and symmetrizing conditions.

# ACs	Minimalizing conditions	Symmetrizing conditions	Classifications	Identification key	Examples
$\#^{\mathrm{s}} = 4$	None	None	$\#^{\mathrm{s}}_{1\mathrm{st}} = 0,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 4$	IV-0-4	Mixed traces
$\#^{\mathrm{s}} = 3$	$[\mathcal{S}^1, \mathcal{S}^{\mathrm{n}}]$	$[\mathcal{S}^1, \mathcal{H}]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 1,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 2$	III-1-2	Unknown
		None	$\#^{\mathrm{s}}_{1\mathrm{st}} = 0,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 4$	III-0-4	Unknown
$\#^{\mathrm{s}} = 2$	$[\mathcal{S}^1, \mathcal{S}^{\mathrm{n}}]$ & $[\mathcal{S}^2, \mathcal{S}^2]$	$[\mathcal{S}^1, \mathcal{H}]$ & $[\mathcal{S}^2, \mathcal{H}]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 2,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 0$	II-2-0	Unknown
		$[\mathcal{S}^2, \dot{\mathcal{S}}^1]$ & $[\mathcal{S}^2, \mathcal{H}]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 1,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 2$	II-1-2b	Unknown
		None	$\#^{\mathrm{s}}_{1\mathrm{st}} = 0,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 4$	II-0-4b	Linear AC
	$[\mathcal{S}^1, \mathcal{S}^{\mathrm{n}}]$ & $[\mathcal{S}^1, \dot{\mathcal{S}}^1]$	$[\dot{\mathcal{S}}^1, H_{\mathrm{P}}]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 1,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 2$	II-1-2a	4dEGB
		None	$\#^{\mathrm{s}}_{1\mathrm{st}} = 0,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 4$	II-0-4a	Unknown
$\#^{\mathrm{s}} = 1$	$[\mathcal{S}^1, \mathcal{S}^1]$, $[\mathcal{S}^1, \dot{\mathcal{S}}^1]$ & $[\dot{\mathcal{S}}^1, \dot{\mathcal{S}}^1]$	$[\dot{\mathcal{S}}^1, \mathcal{H}]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 2,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 0$	I-2-0	GR & $f(\mathcal{H})$
		$[\dot{\mathcal{S}}^1, \ddot{\mathcal{S}}^1]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 1,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 2$	I-1-2b	Unknown
	$[\mathcal{S}^1, \mathcal{S}^1]$, $[\mathcal{S}^1, \dot{\mathcal{S}}^1]$ & $[\mathcal{S}^1, \ddot{\mathcal{S}}^1]$	$[\ddot{\mathcal{S}}^1, \mathcal{H}]$	$\#^{\mathrm{s}}_{1\mathrm{st}} = 1,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 2$	I-1-2a	Cuscuton & QEC
		None	$\#^{\mathrm{s}}_{1\mathrm{st}} = 0,\ \#^{\mathrm{s}}_{2\mathrm{nd}} = 4$	I-0-4	Unknown

An example: Cayley-Hamilton construction

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

$$\mathscr{R}^{\mathrm{I}} \equiv \left\{R^i_i, R^i_j R^j_i, R^i_j R^j_k R^k_i\right\}$$
$$\Pi^{\mathrm{I}} \equiv \left\{\pi^i_i, \pi^i_j \pi^j_i, \pi^i_j \pi^j_k \pi^k_i\right\}$$
$$\mathscr{Q}^{\mathrm{I}} \equiv \left\{R^i_j \pi^j_i, R^i_j \pi^j_k \pi^k_i, R^i_j R^j_k \pi^k_i\right\}$$

$$\mathcal{S}^{\mathrm{I}} = \mathscr{Q}^{\mathrm{I}} - \mathscr{P}^{\mathrm{I}}(N) \approx 0, \quad \mathrm{I} = 1,2,3$$

$$H_{\mathrm{T}}^{(\mathrm{C.H.})} = \int \mathrm{d}^3x \left[\mathscr{H}^{(\mathrm{C.H.})} + N^i \mathcal{H}_i + \lambda^i \pi_i + \lambda \pi + \mu_{\mathrm{I}} \left(\mathscr{Q}^{\mathrm{I}} - \mathscr{P}^{\mathrm{I}}\right)\right],$$

$$\mathscr{H}^{(\mathrm{C.H.})} = \mathscr{H}^{(\mathrm{C.H.})}\left(N, R_{ij}, \pi^{ij}\right)$$

Summary

There are non-GR theories propagating TTDOF’s respecting only spatial covariance.

We find the TTDOF conditions in 2 approaches:

1. (Lagrangian side) spatially covariant gravity with TTDOFs;
2. (Hamiltonian side) TTDOFs with auxiliary constraint(s).

Open questions:

Concrete Lagrangian (so that can be applied in practice)?
Different from/equivalent to GR?
To be tested against the observations?

Thank you for your attention!