Ghosts on the way to a gravity QFT

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a QFT for gravity

- a candidate has been known for a long time (Stelle 1977)
- quantum quadratic gravity (QQG) is a renormalizable and UV complete QFT
  - a new calculation of β-functions: both couplings can be asymptotically free (Buccio, Donoghue, Menezes, Percacci)
- if it works, the implication is that the theory at super Planckian energies has a continuum spacetime description

a stumbling block

- QQG has a ghost, a massive spin-2 partner of the graviton
- ghost: a field with wrong sign kinetic term
- canonical quantization then leads to a negative norm state

problems NOT caused by a negative norm state

- instability: NO
- instability of the classical theory has been traded for a negative norm state of the quantum theory
- all perturbative states have positive energies
- loss of unitarity: NO
- theory still has S-matrix unitarity ($SS^\dagger = 1$)
  i.e. the optical theorem is satisfied

something good about negative norm states

- in QQG, scattering amplitudes of gravitons at super-Planckian energies have bad high energy behavior
- but negative norm states cause cancellations to occur at the level of differential cross sections
- then sufficiently inclusive differential cross sections have good high energy behaviour, due to negative norms

what is the problem?

- negative norms can produce a negative probability via the Born rule, $P = |\langle f|i\rangle|^2/(\langle f|f\rangle\langle i|i\rangle)$
- sign of the norm of a state is called "ghost parity"
- theories that preserve ghost parity:
  probability interpretation exists even with negative norms
- theories that violate ghost parity:
  something more is needed

go back to quantum mechanics

- viewed as 0+1 dimensional quantum field theory
- the quantum gravity problem has already brought focus on some extensions of QM
  - PT-symmetric QM (Mannheim, Bender)
  - Dirac-Pauli quantization (Salvio, Strumia)
- we will follow canonical quantization and see where it leads
- path will sometimes overlap with one or the other of these other approaches

basic picture

- unitary evolution is fundamental to QM
- spectrum of theory and correlation functions in QFT are consequences
- formulation of Born rule to obtain probabilities is a separate aspect of QM
- modification of the latter need not affect the former
- does modification exist and is it unique?

outline

- set up the theory (early collaborator: James Stokes)
- identify two apparent problems
- then show how the following can arise
  - positive spectrum—at weak and strong coupling
  - positive inner product
- obtain wave functions in a coordinate representation
- compare spectra with PT-symmetric QM
- consider complex-conjugate energy eigenvalues

setup Hamiltonian with parameter $\sigma = \pm 1$

- $\sigma = 1$ for normal and $\sigma = -1$ for ghost

$$H_\sigma = \frac{\sigma}{2}(\pi^2 + m^2\phi^2) + \frac{\lambda}{k!}\phi^k$$

$$[\phi, \pi] = i$$

$$\phi = \frac{1}{\sqrt{2m}}(a + a^\dagger), \quad \pi = \frac{\sqrt{m}}{i\sqrt{2}}(a - a^\dagger),$$

$$[a, a^\dagger] = 1$$

Hilbert space

- construction of occupation number basis

$$\text{normal:} \quad a|0\rangle_{+} = 0, \qquad |n\rangle_{+} = \frac{1}{\sqrt{n!}}(a^{\dagger})^{n}|0\rangle_{+}, \quad n \geq 1,$$

$$\text{ghost:} \quad a^{\dagger}|0\rangle_{-} = 0, \qquad |n\rangle_{-} = \frac{1}{\sqrt{n!}}a^{n}|0\rangle_{-}, \quad n \geq 1$$

- σ shows up in the norms of these states

$$_{\sigma}\langle m|n\rangle_{\sigma} = \sigma^{n}\delta_{mn}$$

σ enters in two other ways

- completeness relation

$$\mathbb{1} = \sum_{n\geq 0} \frac{|n\rangle_{\sigma\,\sigma}\langle n|}{{}_\sigma\langle n|n\rangle_\sigma} = \sum_{n\geq 0} \left(|n\rangle_{\sigma\,\sigma}\langle n|\right)\sigma^n,$$

- a general state $|\psi\rangle_\sigma = \sum_{n\geq 0} \psi_{n,\sigma}|n\rangle_\sigma$ has expansion coefficients

$$\psi_{n,\sigma} = ({}_\sigma\langle n|\psi\rangle_\sigma)\sigma^n$$

- there are different but equivalent ways to quantize ghosts
  e.g. Salvio and Strumia 2016

develop matrix notation

- infinite matrix $\mathbf{A}_\sigma$ and infinite column vector $\boldsymbol{\psi}_\sigma$

$$(\mathbf{A}_\sigma)_{mn} = {}_\sigma\langle m|A|n\rangle_\sigma,$$
$$(\boldsymbol{\psi}_\sigma)_n = \psi_{n,\sigma}$$

- also

$$\boldsymbol{\eta}_\sigma = \begin{bmatrix} \sigma^0 & 0 & 0 & \cdots \\ 0 & \sigma^1 & 0 & \cdots \\ 0 & 0 & \sigma^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- $\boldsymbol{\eta}_-$ represents a ghost parity operator

develop matrix notation continued

- operator product $AB \to \mathbf{A}\boldsymbol{\eta}_\sigma\mathbf{B}$
- matrix representation of operator depends on σ, e.g.

$$\boldsymbol{\phi}_\sigma = \begin{bmatrix} 0 & \sigma & 0 & 0 & \cdots \\ \sigma & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sigma\sqrt{3} & \cdots \\ 0 & 0 & \sigma\sqrt{3} & 0 & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

- shall drop the σ subscript and assume (mostly) that $\sigma = -1$

inner product

- the inner product in matrix notation is

$$\langle \psi | \chi \rangle = \boldsymbol{\psi}^\dagger \boldsymbol{\eta} \boldsymbol{\chi} = (\boldsymbol{\eta}\boldsymbol{\psi})^\dagger \boldsymbol{\chi}$$

- a self-adjoint operator satisfies

$$\langle \psi | A\chi \rangle = \langle A\psi | \chi \rangle \quad \rightarrow \quad \mathbf{A}^\dagger \boldsymbol{\eta} = \boldsymbol{\eta} \mathbf{A}$$

- by Hermitian we mean $\mathbf{A}^\dagger = \mathbf{A}$

- $\mathbf{A}$ is self-adjoint $\longleftrightarrow$ $\tilde{\mathbf{A}} \equiv \boldsymbol{\eta}\mathbf{A}$ is Hermitian
  $\mathbf{A}$ is Hermitian $\longleftrightarrow$ $\tilde{\mathbf{A}} \equiv \boldsymbol{\eta}\mathbf{A}$ is self-adjoint

- ghost Hamiltonian **H** is Hermitian

unitary evolution

- translate $|\psi(t)\rangle = \exp(-itH)|\psi\rangle$ into evolution of $\boldsymbol{\psi}(t)$
- the self-adjoint matrix $\tilde{\mathbf{H}}$ appears

$$\boldsymbol{\psi}(t) = \exp(-it\tilde{\mathbf{H}})\boldsymbol{\psi}$$
$$(\boldsymbol{\eta}\boldsymbol{\psi}(t))^\dagger = (\boldsymbol{\eta}\boldsymbol{\psi})^\dagger \exp(it\tilde{\mathbf{H}})$$

- time dependence cancels in $\langle\psi(t)|\chi(t)\rangle = (\boldsymbol{\eta}\boldsymbol{\psi})^\dagger\boldsymbol{\chi}$, as required by unitary evolution

energy eigenstates

- let the energies E_n and the states $|\bar{n}\rangle$ be the eigenvalues and eigenstates of the full Hamiltonian

$$H|\bar{n}\rangle = E_n|\bar{n}\rangle, \quad n = 0, 1, 2, \ldots$$

- translating to matrix notation with $|\bar{n}\rangle \to \boldsymbol{\psi}^{(n)}$ gives

$$\tilde{\mathbf{H}}\boldsymbol{\psi}^{(n)} = E_n\boldsymbol{\psi}^{(n)}$$
$$(\boldsymbol{\eta}\boldsymbol{\psi}^{(n)})^\dagger\tilde{\mathbf{H}} = E_n^*(\boldsymbol{\eta}\boldsymbol{\psi}^{(n)})^\dagger$$

- two types of energy eigenvectors of $\tilde{\mathbf{H}}$, the right- and left-eigenvectors respectively

- E_n must be real except when the norm vanishes

Born rule

- probability for a transition $\boldsymbol{\psi}_i \to \boldsymbol{\psi}_f$
- Born rule from η inner product

$$\Pr(i \to f) = \frac{|(\boldsymbol{\eta}\boldsymbol{\psi}_i)^\dagger \boldsymbol{\psi}_f|^2}{((\boldsymbol{\eta}\boldsymbol{\psi}_i)^\dagger \boldsymbol{\psi}_i)((\boldsymbol{\eta}\boldsymbol{\psi}_f)^\dagger \boldsymbol{\psi}_f)}$$

- gets $\sum_f \Pr(i \to f) = 1$ automatically
- but FAILS to get $0 \le \Pr(i \to f) \le 1$ for any i and f

two apparent problems

1. energy spectrum not guaranteed to be real

$$\tilde{\mathbf{H}}^\dagger \neq \tilde{\mathbf{H}}$$

2. negative norms, e.g. for energy eigenstates

$$\langle \bar{m} | \bar{n} \rangle = (\boldsymbol{\eta} \boldsymbol{\psi}^{(m)})^\dagger \boldsymbol{\psi}^{(n)} \propto (-1)^n \delta_{mn}$$

- let us look for complex energies first
- deal with negative norms after

back to $\tilde{\mathbf{H}}$

- $\tilde{\mathbf{H}}$ obtained from H via $\phi \to \sqrt{1/2m}\,\tilde{\boldsymbol{\phi}}$ and $\pi \to \sqrt{m/2}\,\tilde{\boldsymbol{\pi}}$
- for the quadratic part

$$\frac{\sigma}{2}(\tilde{\boldsymbol{\pi}}^2 + m^2\tilde{\boldsymbol{\phi}}^2) = m\boldsymbol{h}, \qquad \boldsymbol{h} \equiv \begin{bmatrix} 1/2 & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & \cdots \\ 0 & 0 & 5/2 & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

and so we have the same positive-energy (free) spectrum for either $\sigma = \pm 1$

the k-even theories

$$\tilde{\mathbf{H}} = m\boldsymbol{h} + \sigma^{\frac{k}{2}} \frac{\lambda}{(2m)^{\frac{k}{2}} k!} \tilde{\boldsymbol{\phi}}^{k}, \qquad k = 4, 6, 8, \ldots$$

- these theories have real spectra since $\tilde{\mathbf{H}}^{\dagger} = \tilde{\mathbf{H}}$
- they also have conserved ghost parity, $[\tilde{\mathbf{H}}, \boldsymbol{\eta}] = 0$
- all probabilities automatically positive
- negative norms co-exist with probability interpretation

the k-even theories continued

- normal and ghost k-even theories have different $\tilde{\mathbf{H}}$ but they are isospectral!
- full propagators of the two theories differ by a sign
- amplitudes with $2, 6, 10, \ldots$ ghosts differ by a sign
- with only these differences, a ghost theory with ghost parity seems physically equivalent to the corresponding normal theory

the k-odd theories

$$\tilde{\mathbf{H}} = m\boldsymbol{h} + \frac{\lambda}{(2m)^{\frac{k}{2}} k!} \tilde{\boldsymbol{\phi}}^{k} \qquad k = 3, 5, 7, \ldots.$$

- now normal and ghost theories are distinctly different
- normal k-odd theory has well-known problem, spectrum is not bounded from below (at least for λ sufficiently large)
- ghost k-odd theory has $\tilde{\mathbf{H}}^{\dagger} \neq \tilde{\mathbf{H}}$, and so are complex energies an additional problem?
- looks are deceiving — the k-odd ghost theory turns out to enjoy a positive real spectrum for any coupling

numerical study of k-odd ghost theory

- truncate the Hilbert space by truncating the matrix $\tilde{\mathbf{H}}$ to a finite size, then diagonalize
- the number of positive energy eigenvalues grows as the size of $\tilde{\mathbf{H}}$ grows
- e.g. for a matrix size of 600×600 the first 40 eigenvalues are positive for both weak and strong couplings
- complex energies pushed to infinity in infinite size limit

numerical results for $k = 3$ ghost theory

	$\lambda = 1/10$		$\lambda = 10$	
n	$E_n - E_0$	Z_n	$E_n - E_0$	Z_n
0	0.0	0.0006212113	0.0	0.1390183399
1	1.0020710213	−1.000547848	2.4051860435	−1.0512830324
2	2.0061953609	0.0005480361	5.208574457	0.0522332501
3	3.0123535145	−1.881e-7	8.2521064593	−0.0009618532
4	4.0205264034	1.0e-10	11.4766245676	1.17488e-5
5	5.030695361		14.848397294	−1.143e-7
6	6.0428421179		18.3452838701	1.0e-9
7	7.0569487895		21.951494836	
8	8.072997863		25.655134468	
9	9.0909721854		29.4468816584	

- interactions push the energy levels further apart; no merging of levels

Z_n and the full propagator

$$\frac{\langle\bar{0}|T[\phi(t_b)\phi(t_a)]|\bar{0}\rangle}{\langle\bar{0}|\bar{0}\rangle} = \frac{\langle\bar{0}|\phi|\bar{0}\rangle^2}{\langle\bar{0}|\bar{0}\rangle^2} + \sum_{n=1}^{\infty} Z_n D_F(t_b - t_a, E_n - E_0)$$

- D_F is the free Feynman propagator in 0+1d

$$D_F(\tau,\mu) = \frac{1}{2\mu} e^{-i(\mu - i\epsilon)|\tau|}$$

$$Z_n = 2(E_n - E_0)\frac{\langle\bar{0}|\phi|\bar{n}\rangle\langle\bar{n}|\phi|\bar{0}\rangle}{\langle\bar{0}|\bar{0}\rangle\langle\bar{n}|\bar{n}\rangle}$$

- accurate result for propagator with only a small number of terms

new inner product

- consider the G inner product defined by Hermitian matrix $\mathbf{G}$

$$\langle \psi | \chi \rangle_G = \boldsymbol{\psi}^\dagger \mathbf{G} \boldsymbol{\chi} = (\mathbf{G}\boldsymbol{\psi})^\dagger \boldsymbol{\chi}$$

- also write as

$$\langle \psi | \chi \rangle_G = (\boldsymbol{\eta}\tilde{\mathbf{G}}\boldsymbol{\psi})^\dagger \boldsymbol{\chi} \quad \text{with} \quad \tilde{\mathbf{G}} \equiv \boldsymbol{\eta}\mathbf{G}$$

- for G inner product to be preserved in time we need

$$[\tilde{\mathbf{G}}, \tilde{\mathbf{H}}] = 0$$

- thus the energy eigenstates are also eigenstates of $\tilde{\mathbf{G}}$

what is $\tilde{\mathbf{G}}$?

- we want negative norms to become positive norms, so

$$\tilde{\mathbf{G}}\boldsymbol{\psi}^{(n)} = (-1)^n \boldsymbol{\psi}^{(n)}$$

- explicit realization

$$\tilde{\mathbf{G}} = \sum_n \frac{\boldsymbol{\psi}^{(n)}(\boldsymbol{\eta}\boldsymbol{\psi}^{(n)})^\dagger}{|(\boldsymbol{\eta}\boldsymbol{\psi}^{(n)})^\dagger \boldsymbol{\psi}^{(n)})|}$$

- this is a nontrivial matrix

in addition...

$$\tilde{\mathbf{G}}^2 = \sum_n \frac{\boldsymbol{\psi}^{(n)}(\boldsymbol{\eta}\boldsymbol{\psi}^{(n)})^\dagger}{(\boldsymbol{\eta}\boldsymbol{\psi}^{(n)})^\dagger\boldsymbol{\psi}^{(n)})} = 1$$

- $\tilde{\mathbf{G}}$ is self-adjoint wrt both the G and η inner product, as is $\tilde{\mathbf{H}}$

$$\tilde{\mathbf{G}}^\dagger \mathbf{G} = \mathbf{G}\tilde{\mathbf{G}}$$

$$\tilde{\mathbf{H}}^\dagger \mathbf{G} = \mathbf{G}\tilde{\mathbf{H}}$$

- $\mathbf{G}$ is a **positive-definite** Hermitian matrix

recovering the Dirac norm

- the G inner product is trivial for the energy eigenstates

$$\langle \bar{m}|\bar{n}\rangle_G = (\mathbf{G}\boldsymbol{\psi}^{(m)})^\dagger \boldsymbol{\psi}^{(n)} \propto \delta_{mn}$$

- write a general state in terms of the energy eigenstates

$$|\psi\rangle = \sum_{n\geq 0} \tilde{\psi}_n |\bar{n}\rangle$$

- thus define $\tilde{\boldsymbol{\psi}}$ and $\tilde{\boldsymbol{\chi}}$ so that the G inner product becomes

$$\langle \psi|\chi\rangle_G = \tilde{\boldsymbol{\psi}}^\dagger \tilde{\boldsymbol{\chi}}$$

sensible Born rule

- first determine the energy eigenstates of the full Hamiltonian
- use this as a basis in which to obtain the $\tilde{\boldsymbol{\psi}}$'s, then

$$\Pr(i \to f) = \frac{|\tilde{\boldsymbol{\psi}}_f^\dagger \tilde{\boldsymbol{\psi}}_i|^2}{(\tilde{\boldsymbol{\psi}}_i^\dagger \tilde{\boldsymbol{\psi}}_i)(\tilde{\boldsymbol{\psi}}_f^\dagger \tilde{\boldsymbol{\psi}}_f)}$$

- alternatively, occupation number basis can be used

$$\Pr(i \to f) = \frac{|(\mathbf{G}\boldsymbol{\psi}_i)^\dagger \boldsymbol{\psi}_f|^2}{((\mathbf{G}\boldsymbol{\psi}_i)^\dagger \boldsymbol{\psi}_i)((\mathbf{G}\boldsymbol{\psi}_f)^\dagger \boldsymbol{\psi}_f)}$$

finding observables

- the eigenvalue equation $A|\lambda\rangle = \lambda|\lambda\rangle$ when translated to matrix notation becomes $\tilde{\mathbf{A}}\boldsymbol{\psi}^{\lambda} = \lambda\boldsymbol{\psi}^{\lambda}$
- Hermitian $\tilde{\mathbf{A}}$ gives real eigenvalues
  then $\mathbf{A}$ is self-adjoint
- thus self-adjoint operators can be observables
- Hamiltonian H is Hermitian and is thus an exception

the coordinate representation

- matrix form of the commutation relation $[q, p] = i$ is

$$\mathbf{q}\eta\mathbf{p} - \mathbf{p}\eta\mathbf{q} = i\eta,$$
$$\text{or} \quad \tilde{\mathbf{q}}\tilde{\mathbf{p}} - \tilde{\mathbf{p}}\tilde{\mathbf{q}} = i.$$

- two sets of solutions

$$\tilde{\mathbf{q}} = \boldsymbol{\phi} \quad \tilde{\mathbf{p}} = \boldsymbol{\pi}$$
$$\tilde{\mathbf{q}} = i\tilde{\boldsymbol{\phi}} \quad \tilde{\mathbf{p}} = i\tilde{\boldsymbol{\pi}}$$

- these are Hermitian as desired
  second set is Dirac-Pauli choice

coordinate eigenvectors

- the eigenvalues of the matrix $i\tilde{\boldsymbol{\phi}}$ are a set of real positions x

$$(i\tilde{\boldsymbol{\phi}})\boldsymbol{\psi}^{(x)} = x\boldsymbol{\psi}^{(x)}$$

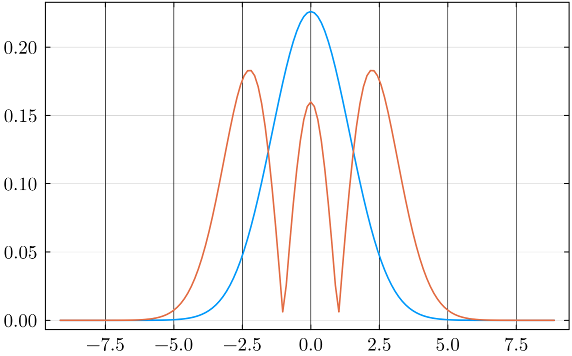
- eigenvectors can be used to construct the wave functions (functions of x) for the exact energy eigenstates $|\bar{n}\rangle$

$$\langle x|\bar{n}\rangle = (\boldsymbol{\eta}\boldsymbol{\psi}^{(x)})^{\dagger}\boldsymbol{\psi}^{(n)}$$

- $\langle p|\bar{n}\rangle$ obtained in similar way

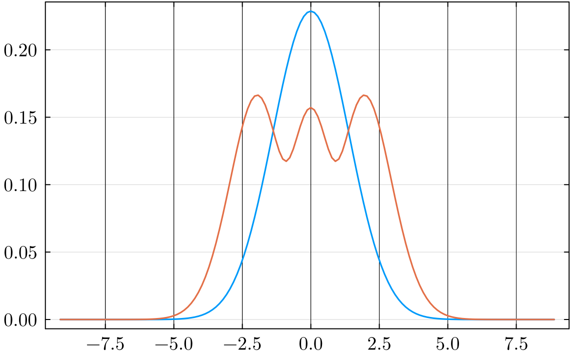
wave functions at zero coupling

- $|\langle x|\bar{0}\rangle|$ and $|\langle x|\bar{2}\rangle|$ are the same as $|\langle p|\bar{0}\rangle|$ and $|\langle p|\bar{2}\rangle|$



coordinate wave functions at $\lambda = 1$

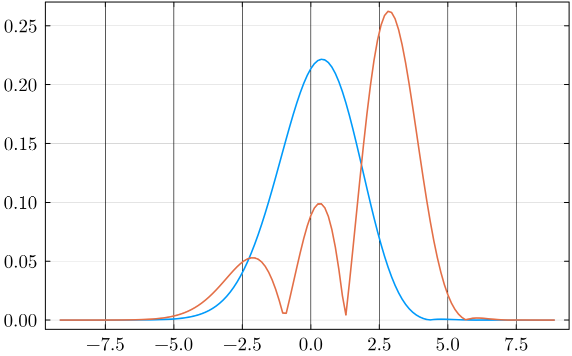
- $|\langle x|\bar{0}\rangle|$ and $|\langle x|\bar{2}\rangle|$



- interactions cause wave functions to become more localized

momentum wave functions at $\lambda = 1$

- $|\langle p|\bar{0}\rangle|$ and $|\langle p|\bar{2}\rangle|$



- normal parity (implemented by the action of $\boldsymbol{\eta}$) is explicitly broken due to interactions

effect of the G inner product

- G inner product keeps $|\langle x|\bar{n}\rangle_G| = |\langle x|\bar{n}\rangle|$
- η inner product in coord basis $\langle x'|x\rangle \propto \delta_{-x,x'}$ has negative eigenvalues (just like $\langle n'|n\rangle$ or $\langle \bar{n}'|\bar{n}\rangle$)
- G inner product in coord basis $\langle x'|x\rangle_G$ has no negative eigenvalues and when the coupling vanishes it goes to the standard result $\langle x'|x\rangle_G = \delta_{x,x'}$

the PT connection

- the determination of a sensible inner product using the eigenstates of a non-Hermitian Hamiltonian arises in the study of PT-symmetric QM

- those studies use a complex extension of ordinary quantum mechanics via the construction of a complex coordinate-space representation

- ghost theories instead start with a Hermitian Hamiltonian and a canonical quantization, and it is the effect of negative norms that leads to the non-Hermitian $\tilde{\mathbf{H}}$

canonical quantization of PT theories

- a PT non-Hermitian Hamiltonian has positive kinetic terms and so a $\sigma = 1$ canonical quantization can proceed
- but the resulting norm $\langle n' | n \rangle$ is not preserved under time evolution
- major theme of PT studies is to find a positive definite inner product
- both the η and the G inner products carry over to the PT theory

compare theories

- the k-odd ghost theory is transformed into a PT theory via $\phi \to i\phi$ and $\pi \to i\pi$
- repeat numerical analysis for $\sigma = 1$ quantization
  - spectrum matches ghost theory
  - full propagator is the negative of the ghost propagator
- not surprising—expect that general amplitudes in the two theories will only differ by powers of i
- it then appears that the two QFTs are equivalent

more examples

- numerical spectra for various PT theories are given in the literature, such as for

$$H = \pi^2 + \phi^n (i\phi)^\varepsilon \qquad n = 2, 4, 6, \ldots$$

- when $\varepsilon = 1$ the corresponding ghost theories are

$$H = -\pi^2 + \phi^k \qquad k = 3, 5, 7, \ldots$$

- the PT and ghost spectra match

- the same is true for ($\varepsilon = 1$)

$$\begin{aligned} \text{PT theory :} \quad & H = \pi^m + \phi^2 (i\phi)^\varepsilon \quad m = 4, 6, 8, \ldots \\ \text{ghost theory :} \quad & H = (-1)^{\frac{m}{2}} \pi^m + \phi^3 \end{aligned}$$

comments

- with $\varepsilon = 1$ these particular PT theories have Stokes wedges that continue to include the real axis
- only these PT theories are connected to the ghost theories
- we can also carry over the construction of the coordinate representation to the PT theory, via $\boldsymbol{\phi}\boldsymbol{\psi}^{(x)} = x\boldsymbol{\psi}^{(x)}$
- again real values of x and normalizable wave-functions are obtained

consider spectra with complex-conjugate pairs

- occurs for k-odd ghost theory when $\tilde{\mathbf{H}}$ is truncated to finite size and when the coupling is sufficiently large
- in a 4d QFT, propagator poles at complex conjugate locations can appear due to 1-loop correction
- the meaning of this when this happens for a ghost propagator is a subject of great debate

complex-conjugate pairs

- such a pair of states labelled by $(p, p+1)$ have vanishing norms
$$(\boldsymbol{\eta}\boldsymbol{\psi}^{(p)})^\dagger \boldsymbol{\psi}^p = 0, \quad (\boldsymbol{\eta}\boldsymbol{\psi}^{(p+1)})^\dagger \boldsymbol{\psi}^{p+1} = 0$$
- there are instead off-diagonal inner products
$$(\boldsymbol{\eta}\boldsymbol{\psi}^{(p)})^\dagger \boldsymbol{\psi}^{p+1} = e^{i\theta_p}, \quad (\boldsymbol{\eta}\boldsymbol{\psi}^{(p+1)})^\dagger \boldsymbol{\psi}^{p} = e^{-i\theta_p}$$
- the set of all states is now $\{n\} = \{q\} \cup \{p\}$, where the q states are as before
$$(\boldsymbol{\eta}\boldsymbol{\psi}^{(q)})^\dagger \boldsymbol{\psi}^q = (-1)^q, \quad q = 0, 1, 2, 3, \ldots$$

define α and β states to have diagonal norms

$$\boldsymbol{\psi}_{\alpha}^{(p)} = e^{i\alpha_p}\boldsymbol{\psi}^{(p+1)} + e^{-i\alpha_p}\boldsymbol{\psi}^{(p)}$$
$$\boldsymbol{\psi}_{\beta}^{(p)} = e^{i\beta_p}\boldsymbol{\psi}^{(p+1)} + e^{-i\beta_p}\boldsymbol{\psi}^{(p)}$$

- by setting $\beta_p = \pi/2 - \theta_p - \alpha_p$, the α- and β-states are orthogonal and

$$\boldsymbol{\psi}_{\alpha}^{(p)\dagger}\boldsymbol{\eta}\boldsymbol{\psi}_{\alpha}^{(p)} = 2\cos(2\alpha_p + \theta_p), \quad \boldsymbol{\psi}_{\beta}^{(p)\dagger}\boldsymbol{\eta}\boldsymbol{\psi}_{\beta}^{(p)} = -2\cos(2\alpha_p + \theta_p)$$

- one and only one of these norms is negative

remaining freedom

- we choose α_p such that the state $|\bar{\alpha}_p\rangle$ has no overlap with $\phi|\bar{0}\rangle$, that is

$$\langle\bar{\alpha}_p|\phi|\bar{0}\rangle = \boldsymbol{\psi}_\alpha^{(p)\dagger}\boldsymbol{\phi}\boldsymbol{\psi}^{(0)} = 0$$

- then the state $|\bar{\alpha}_p\rangle$ does not contribute to the spectral representation of the propagator

- in a higher dimensional QFT another implication of $\langle\bar{\alpha}_p|\phi|\bar{0}\rangle = 0$ is that the state $|\bar{\alpha}_p\rangle$ is not an asymptotic state

- asymptotic states are those that can participate in scattering experiments

positive inner product?

- require $[\tilde{\mathbf{G}}, \tilde{\mathbf{H}}] = 0$ but $\tilde{\mathbf{H}}$ is not diagonal in the $(\bar{\alpha}_p, \bar{\beta}_p)$ subspace
- there is only one sign to choose $\tilde{\mathbf{G}}(\boldsymbol{\psi}_\alpha^{(p)}, \boldsymbol{\psi}_\beta^{(p)}) = \pm(\boldsymbol{\psi}_\alpha^{(p)}, \boldsymbol{\psi}_\beta^{(p)})$
- choose it so that the G norm of the β-state is positive
- now only the α-states, the states that are not asymptotic states, have negative G-norm

expression for $\tilde{\mathbf{G}}$

$$\tilde{\mathbf{G}} = \sum_{\{q\}} \frac{\boldsymbol{\psi}^{(q)}(\boldsymbol{\eta}\boldsymbol{\psi}^{(q)})^\dagger}{|(\boldsymbol{\eta}\boldsymbol{\psi}^{(q)})^\dagger\boldsymbol{\psi}^{(q)})|} + \sum_{\{p\}} \left[-\frac{\boldsymbol{\psi}_\alpha^{(p)}(\boldsymbol{\eta}\boldsymbol{\psi}_\alpha^{(p)})^\dagger}{|(\boldsymbol{\eta}\boldsymbol{\psi}_\alpha^{(p)})^\dagger\boldsymbol{\psi}_\alpha^{(p)}|} + \frac{\boldsymbol{\psi}_\beta^{(p)}(\boldsymbol{\eta}\boldsymbol{\psi}_\beta^{(p)})^\dagger}{|(\boldsymbol{\eta}\boldsymbol{\psi}_\beta^{(p)})^\dagger\boldsymbol{\psi}_\beta^{(p)}|} \right]$$

Born rule again

- an arbitrary state in the space of asymptotic states is

$$|\psi\rangle = \sum_{\{q\}} \tilde{\psi}_q |\bar{q}\rangle + \sum_{\{p\}} \tilde{\psi}_p |\bar{\beta}_p\rangle$$

- from this, form a column vector $\tilde{\boldsymbol{\psi}}$
- as before, the G norm in terms of such column vectors is $\langle\psi|\chi\rangle_G = \tilde{\boldsymbol{\psi}}^\dagger \tilde{\boldsymbol{\chi}}$
- this construction gives standard Born rule

full propagator (contributions from c.c. pairs)

$$\sum_{\{q\}}(\ldots) + \sum_{\{p\}} \Big(Z_p D_F(t_b - t_a, E_p - E_0) + Z_p^* D_F(t_b - t_a, E_p^* - E_0) \Big)$$

- there is also a sum rule

$$\sum_{q=1}^{\infty} Z_q + \sum_{\{p\}} (Z_p + Z_p^*) = 1$$

$$Z_p + Z_p^* = 2(E_p^r - E_0) \frac{\langle \bar{0} | \phi | \bar{\beta}_p \rangle \langle \bar{\beta}_p | \phi | \bar{0} \rangle}{\langle \bar{0} | \bar{0} \rangle \langle \bar{\beta}_p | \bar{\beta}_p \rangle}$$

- α-states do not appear

lesson for 3+1d ghost QFT

- complex conjugate poles appear in the ghost propagator due to 1-loop correction
- Kubo and Kugo argue that an asymptotic state remains, and that it is a ghost
- this is the analog of our β-state
- if the G inner product can be generalized to higher dimensions then we can recover standard Born rule

conclusion

- is there anything in principle that blocks the extension from 0+1d to 3+1d QFT?
  - e.g. continuous spectra or degenerate states?
- if not then opens way for QQG to give a consistent continuum spacetime description of super Planckian energies
- QCD is already showing what a UV complete QFT looks like — quantum gravity may look like something similar
- did we give up on quantum field theory too early?