Physical running of couplings in quadratic gravity

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Quantum Gravity and Cosmology 2024,
Shanghai, 01/06/2024

Based on

D. Buccio, J. F. Donoghue and R. P.
"Amplitudes and renormalization group techniques: A case study,"
Phys. Rev. D **109** (2024) no.4, 045008 [arXiv:2307.00055 [hep-th]].

D. Buccio, J. F. Donoghue, G. Menezes and R.P.
"Physical running of couplings in quadratic gravity,"
PRL, in print [arXiv:2403.02397 [hep-th]].

Various flavors of RG running

Physical running. Define the coupling in terms of the scattering amplitude at some particular momentum $p = E$. Changing E changes the value of the coupling.

$\mu-$**running.** In perturbation theory using dimreg or cutoff regularization one has to introduce a parameter μ to preserve dimensions, e.g. in $\log(p^2/\mu^2)$. Taking the derivative of the coupling with respect to μ defines another kind of RG.

Non-perturbative RG. One studies the dependence of the couplings in the quantum effective action on a UV cutoff (Wilsonian RG) or IR cutoff (FRG).

When do they give the same results?

Shift-invariant scalar

$$\mathcal{L} = -\frac{Z_1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_2\Box\phi\Box\phi - \frac{1}{4}Z_2^2 g(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi)$$

with $Z_2 = \frac{Z_1}{m^2}$. ($[Z_1] = [g] = 0$, $[Z_2] = -2$)

Characteristic scales:

- ghost mass m
- interaction scale: $m/\sqrt[4]{g}$

In order for ghosts to be propagating and weakly coupled need $g \ll 1$

Energy domains

- $E < m$ low energy regime: only massless particles propagate and are weakly coupled; massive ghosts do not propagate
- $m < E < m/\sqrt[4]{g}$: intermediate energy regime; also ghosts propagate and are weakly coupled
- $m/\sqrt[4]{g} < E$ high energy regime; apparently strongly interacting

2-point function

$$i\frac{3}{2}\frac{1}{Z_1}p^2\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+\log 4\pi-\gamma-\log\frac{m^2}{\mu^2}+\frac{7}{6}+O(\epsilon)\right)$$

No renormalization of Z_2

There is μ-running of Z_1 but no physical running.

General 4 point amplitude

$$\frac{5g^2\left(s^2+t^2+u^2\right)}{64\pi^2 m^4\epsilon}+\frac{g^2}{5760\pi^2 m^8}\left\{\frac{m^4}{s^2}\left[-6m^4\left(s^2+t^2+u^2\right)+3sm^2\left(-31s^2+9\left(t^2+u^2\right)\right)\right.\right.$$

$$\left.+2s^2\left((352-195\gamma_E)s^2-(15\gamma_E-37)\left(t^2+u^2\right)\right)\right]$$

$$+6s^{-1/2}m^4\sqrt{4m^2-s}[16m^4(6s^2+t^2+u^2)-8sm^2(16s^2+t^2+u^2)+s^2(41s^2+t^2+u^2)]\,\mathrm{arccot}\sqrt{\frac{4m^2}{s}-1}$$

$$+3s^2\left(41s^2+t^2+u^2\right)\log\left(-\frac{m^2}{s}\right)$$

$$+\frac{6(s-m^2)^3}{s^3}\log\left(\frac{m^2}{m^2-s}\right)\left[m^4\left(s^2+t^2+u^2\right)-2sm^2\left(-9s^2+t^2+u^2\right)+s^2\left(41s^2+t^2+u^2\right)\right]$$

$$+(\text{same with } u\to s\to t)+(\text{samewith } t\to u\to s)$$

$$\left.+450m^4\left(s^2+t^2+u^2\right)\log\left(\frac{4\pi\mu^2}{m^2}\right)\right\}$$

Low energy

From EFT argument we expect to generate new interactions

$$\begin{aligned}\mathcal{L}_6 &= \frac{g_6}{4M^6}\partial_\mu\phi\partial^\mu\phi\Box\partial_\nu\phi\partial^\nu\phi + \frac{g_6'}{4M^6}\partial_\mu\phi\partial_\nu\phi\Box\partial^\mu\phi\partial^\nu\phi \\ \mathcal{L}_8 &= -\frac{g_8}{4M^8}\partial_\mu\phi\partial^\mu\phi\Box^2\partial_\nu\phi\partial^\nu\phi - \frac{g_8'}{4M^8}\partial_\mu\phi\partial_\nu\phi\Box^2\partial^\mu\phi\partial^\nu\phi\end{aligned}$$

Indeed defining

$$g(\mu) = g_B - \frac{5g^2m^4}{32\pi^2M^4}\left[\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{4\pi\mu^2}{m^2}\right) + \frac{11}{30}\right]$$

the amplitude becomes

low energy amplitude

$$
\begin{aligned}
&-\frac{g}{2m^4}(s^2+t^2+u^2)\\
&+\frac{g_6}{2m^6}(s^3+t^3+u^3)+\frac{g_6'}{4m^6}(s^2t+s^2u+t^2u+t^2s+u^2s+u^2t)\\
&+\frac{g_8(E)}{m^8}(s^4+t^4+u^4)+\frac{g_8'(E)}{2m^8}(s^2t^2+s^2u^2+t^2u^2)\\
&+\frac{g^2}{1920\pi^2m^8}\left[41s^4\log\left(\frac{-s}{E^2}\right)+41t^4\log\left(\frac{-t}{E^2}\right)+41u^4\log\left(\frac{-u}{E^2}\right)\right.\\
&+s^2(t^2+u^2)\log\left(\frac{-s}{E^2}\right)+t^2(s^2+u^2)\log\left(\frac{-t}{E^2}\right)+u^2(t^2+s^2)\log\left(\frac{-u}{E^2}\right)
\end{aligned}
$$

with $g_6=-\frac{53g^2}{384\pi^2}$, $g_6'=-\frac{7g^2}{516\pi^2}$,
$g_8(E)=\frac{79g^2}{1920\pi^2}+\frac{41g^2}{960\pi^2}\log(E^2/m^2)$, $g_8'(E)=\frac{3g^2}{320\pi^2}+\frac{1g^2}{480\pi^2}\log(E^2/m^2)$

Low energy physical beta functions

$$
\begin{aligned}
\beta_g &= 0 \\
\beta_{g_6} &= 0 \\
\beta_{g'_6} &= 0 \\
\beta_{g_8} &= \frac{41 g^2}{480\pi^2} \\
\beta_{g'_8} &= \frac{g^2}{240\pi^2}
\end{aligned}
$$

To be compared with

the μ-beta function

$$\mu\frac{\partial g(\mu)}{\partial \mu} = \frac{5g^2}{16\pi^2}$$

and the low energy FRG

$$\beta_g = \frac{5(Z_1 + 2k^2/m^2)}{32\pi^2(Z_1 + k^2/m^2)^3}\frac{g^2k^4}{M^4} \to \frac{5g^2}{32\pi^2}\frac{k^4}{m^4}$$

that indeed goes to zero in the limit $k \to 0$

High energy amplitude

$$\bar{g}(E) = g + \frac{5g^2}{32\pi^2}\left[\log\left(\frac{E^2}{m^2}\right) - \frac{17}{30}\right]$$

higher derivative terms cancel out

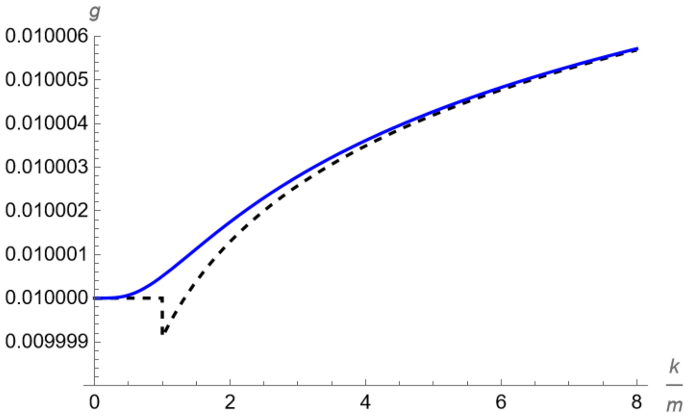
$$\begin{aligned}
&-\frac{\bar{g}(E)}{2m^4}(s^2+t^2+u^2) \\
&+\frac{\bar{g}^2}{192\pi^2 m^4}\left[\log\left(\frac{-s}{E^2}\right)(13s^2+t^2+u^2)\right. \\
&\qquad +\log\left(\frac{-t}{E^2}\right)(s^2+13t^2+u^2) \\
&\qquad \left. +\log\left(\frac{-u}{E^2}\right)(s^2+t^2+13u^2)\right]
\end{aligned}$$

High energy physical beta function

$$\beta_{\bar{g}} = \frac{5\bar{g}^2}{16\pi^2}$$

agrees with the μ-beta function and with the FRG

Matching high and low energy



no physical interpretation in terms of scattering

High energy puzzle

Theory is asymptotically free for $g < 0$

Still it seems to become strongly coupled for $E > m/\sqrt[4]{g}$

What is the meaning of asymptotic freedom in this case?

Asymptotic states

Cancellations at tree level between the contributions of massless particles and ghosts in the inclusive cross sections

B. Holdom, [arXiv:2303.06723 [hep-th]]

Verified also at one loop (D. Buccio)

But why can one not consider exclusive cross sections?

The free theory that one is asymptoting to (dipole ghost) does not have propagating d.o.f.

N. N. Bogolubov, A. A. Logunov, A. I. Oksak, I. T. Todorov, General Principles of Quantum Field Theory, Springer

V.O. Rivelles, Triviality of higher derivative theories, Phys.Lett.B 577 (2003) 137-142, arXiv: hep-th/0304073 [hep-th]

Lessons from this example

- physical running of g only defined in asymptotic regions.
- in the low energy EFT at one loop the coupling g does not run but there are higher order operators with 6 and 8 derivatives, some of which exibit physical running
- higher dimension operators disappear above the mass threshold
- μ-running agrees with physical running far above the mass threshold
- FRG running agrees with physical running far above and far below the mass threshold
- power law running seen in the FRG is an aspect of threshold behavior

Quadratic gravity

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\Big[-2Z\Lambda + ZR - \frac{1}{2\lambda}C^2 - \frac{1}{\xi}R^2 + \frac{1}{\rho}E\Big], \\
&= \int d^4x\,\sqrt{-g}\left[-2Z\Lambda + ZR - \frac{1}{2\lambda}\left(C^2 - \frac{2\omega}{3}R^2 + 2\theta E\right)\right]
\end{aligned}
$$

$$
Z = \frac{1}{2}m_P^2 = \frac{1}{16\pi G}\,,
$$

Note: $S_E = -S_L$

This theory is renormalizable

K. S. Stelle,
"Renormalization of Higher Derivative Quantum Gravity,"
Phys. Rev. D **16** (1977), 953-969

It propagates a massless graviton, a massive spin 2 ghost and a massive (non-ghost) spin 0.

The massive spin 2 is a tachyon for $\lambda < 0$ and the massive spin 0 is a tachyon for $\xi > 0$.

Maybe the issue of the ghost can be circumvented

D. Anselmi and M. Piva, JHEP 05 (2018), 027 [arXiv:1803.07777 [hep-th]].
A. Salvio, Front. in Phys. 6, 77 (2018) [arXiv:1804.09944 [hep-th]].
J. F. Donoghue and G. Menezes, Nuovo Cim. C 45, no.2, 26 (2022) [arXiv:2112.01974 [hep-th]].

L. Buoninfante, JHEP 12 (2023), 111 [arXiv:2308.11324 [hep-th]].

Linearization

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$S^{(2)}(\bar{g}, h) = \int d^4x \sqrt{|\bar{g}|} h_{\alpha\beta} \mathcal{H}^{\alpha\beta,\gamma\delta} h_{\gamma\delta} \ .$$

One can choose the gauge such that

$$\mathcal{H} = \bar{\Box}^2 \mathbb{K} + \mathbb{J}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \mathbb{L}^\mu \bar{\nabla}_\mu + \mathbb{W}$$

$$\mathbb{K} = \frac{1}{4\lambda} \mathbb{P}_{\text{tl}} + \frac{9}{4(3\xi - 2\lambda)} \mathbb{P}_{\text{tr}}$$

where $\mathbb{P}_{\text{tr}}^{\alpha\beta,\gamma\delta} = \frac{1}{4} \bar{g}^{\alpha\beta} \bar{g}^{\gamma\delta}$, $\mathbb{P}_{\text{tl}} = \mathbb{I} - \mathbb{P}_{tr}$

$\mathbb{K}$ is a tensorial wave function renormalization constant that gives different weights to the spin-2 and spin-0 components of h. We canonically normalize the fields by redefining $h \to \sqrt{\mathbb{K}^{-1}}h$, so that the action can be rewritten as

$$S^{(2)} = \int d^4x \sqrt{|\bar{g}|} h_{\alpha\beta} \mathcal{O}^{\alpha\beta,\gamma\delta} h_{\gamma\delta} \ ,$$

where

$$\mathcal{O} = \bar{\Box}^2 \mathbb{I} + \mathbb{V}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \mathbb{N}^\mu \bar{\nabla}_\mu + \mathbb{U} \ ,$$

and $\mathbb{V} = \sqrt{\mathbb{K}^{-1}} \mathbb{J} \sqrt{\mathbb{K}^{-1}}$ etc.

$\mathbb{V} \sim (\bar{R}, m_P^2)$, $\mathbb{N} \sim \bar{\nabla}\bar{R}$, $\mathbb{U} \sim (\bar{R}^2, \bar{\nabla}^2 \bar{R}, m_P^2 \bar{R}, m_P^2 \Lambda)$.

Different ways of using the BF method

- choose a particular background (e.g. a sphere)
- the background is a small perturbation of flat space
- the background is a generic metric

Second method used by

J. Julve, M. Tonin, Nuovo Cim. B **46** (1978) 137.

Third method used by

E.S. Fradkin, A.A. Tseytlin, Phys. Lett. B **104** (1981) 377; Nucl. Phys. B **201** (1982) 469.

I.G. Avramidi, A.O. Barvinski, Phys. Lett. **159 B**, 269 (1985).

and everybody else since then

The logarithmic divergences or $1/\epsilon$ poles are proportional to the heat kernel coefficient
(A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. **119**, 1-74 (1985).)

$$\frac{1}{32\pi^2}\int d^4x\,\mathrm{tr}\Big[\frac{\mathbb{I}}{90}\left(\bar{R}^2_{\rho\lambda\sigma\tau}-\bar{R}^2_{\rho\lambda}+\frac{5}{2}\bar{R}^2\right)+\frac{1}{6}\mathbb{R}_{\rho\lambda}\mathbb{R}^{\rho\lambda}$$
$$-\frac{\bar{R}_{\rho\lambda}\mathbb{V}^{\rho\lambda}-\frac{1}{2}\bar{R}\mathbb{V}^{\rho}{}_{\rho}}{6}+\frac{\mathbb{V}_{\rho\lambda}\mathbb{V}^{\rho\lambda}+\frac{1}{2}\mathbb{V}^{\rho}{}_{\rho}\mathbb{V}^{\lambda}{}_{\lambda}}{24}-\mathbb{U}\Big]\,,$$

where $\mathbb{R}_{\rho\lambda}=[\nabla_\rho,\nabla_\lambda]$ acting on symmetric tensors.

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\omega = -\frac{1}{(4\pi)^2}\frac{25 + 1098\omega + 200\omega^2}{60}\lambda$$

$$\beta_\theta = \frac{1}{(4\pi)^2}\frac{7(56 - 171\theta)}{90}\lambda$$

[I.G. Avramidi, A.O. Barvinski, Phys. Lett. **159 B**, 269 (1985).]

AF in a subset of first quadrant, so spin 0 is a tachyon.

Beta functions confirmed by several other calculations, also using the FRG in various approximations.

[G. de Berredo-Peixoto and I. L. Shapiro, Phys. Rev. D **71** (2005), 064005 [arXiv:hep-th/0412249 [hep-th]].]
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[D. Benedetti, P. F. Machado, F. Saueressig, Mod. Phys. Lett. A **24** (2009) 2233 [arXiv:0901.2984 [hep-th]]
[M. Niedermaier, Nucl. Phys. B833, 226-270 (2010).]

[G. Narain and R. Anishetty, J. Phys. Conf. Ser. **405** (2012), 012024 [arXiv:1210.0513 [hep-th]].]
[K. Groh, S. Rechenberger, F. Saueressig, O. Zanusso, PoS EPS **-HEP2011** (2011) 124 [arXiv:1111.1743 [hep-th]].]
[N. Ohta, R.P. Class. Quant. Grav. **31** 015024 (2014); arXiv:1308.3398]

[K. Falls, N. Ohta and R. Percacci, Phys. Lett. B **810** (2020), 135773 [arXiv:2004.04126 [hep-th]].]
[S. Sen, C. Wetterich, M. Yamada, JHEP 03 (2022) 130, arXiv:2111.04696 [hep-th]

Gravity/QCD analogy

- weakly coupled in IR limit
- AF in the UV limit
- strongly coupled in intermediate regime

B. Holdom and J. Ren, “QCD analogy for quantum gravity,” Phys. Rev. D **93** (2016) no.12, 124030 [arXiv:1512.05305 [hep-th]].

A. Salvio and A. Strumia, Agravity, JHEP 06 (2014) 080, arXiv: 1403.4226 [hep-ph]

Einstein–Hilbert GFP

Expanding $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$S = \frac{1}{G}\int d^d x \left[(\partial h)^2 + h(\partial h)^2 + h^2(\partial h)^2 + \dots\right]$$
$$+\frac{1}{\lambda}\int d^d x \left[(\Box h)^2 + h(\Box h)^2 + h^2(\Box h)^2 + \dots\right]$$

then rescaling $h \to \sqrt{G}\, h$

$$S = \int d^d x \left[(\partial h)^2 + \sqrt{G}\, h(\partial h)^2 + G\, h^2(\partial h)^2 + \dots\right]$$
$$+\frac{G}{\lambda}\int d^d x \left[(\Box h)^2 + \sqrt{G}\, h(\Box h)^2 + G\, h^2(\Box h)^2 + \dots\right]$$

GFP for $\lambda \neq 0$ or $\lambda \to \infty$

Stelle GFP

Expanding $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\begin{aligned} S &= \frac{1}{G}\int d^dx \left[(\partial h)^2 + h(\partial h)^2 + h^2(\partial h)^2 + \ldots\right] \\ &+ \frac{1}{\lambda}\int d^dx \left[(\Box h)^2 + h(\Box h)^2 + h^2(\Box h)^2 + \ldots\right] \end{aligned}$$

rescaling $h \to \sqrt{\lambda}\, h$

$$\begin{aligned} S &= \frac{\lambda}{G}\int d^dx \left[(\partial h)^2 + \sqrt{G}\, h(\partial h)^2 + G\, h^2(\partial h)^2 + \ldots\right] \\ &+ \int d^dx \left[(\Box h)^2 + \sqrt{\lambda}\, h(\Box h)^2 + \lambda\, h^2(\Box h)^2 + \ldots\right] \end{aligned}$$

GFP for $G \neq 0$ or even $G \to \infty$

Summary

- EH FP describes gravity in the IR
- Stelle FP (FP_1) possible UV completion
- there may be other UV completions related to nontrivial FP's

Important questions:

- what is the physics at the UV FP (either Gaussian or not)?
- can we flow from the UV fixed point to the EH FP in the IR?

Physical running in QG

Do the beta functions reflect the behavior of the scattering amplitude?

For that we return to the second way of using the BF method

Remember

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Expand the background field

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$$

Then

$$\begin{aligned} \mathcal{O} \quad &\equiv \quad \Box^2 \mathbb{I} + \mathcal{D}^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma + \mathcal{C}^{\mu\nu\rho}\partial_\mu\partial_\nu\partial_\rho \\ &+ \quad \mathcal{V}^{\mu\nu}\partial_\mu\partial_\nu + \mathcal{N}^\mu\partial_\mu + \mathcal{U} \ , \end{aligned}$$

where $\mathcal{D}$, $\mathcal{C}$, $\mathcal{V}$, $\mathcal{N}$, $\mathcal{U}$ are polynomials in $\bar{h}$

The physical running of λ and ξ can be read off the two point function, which from terms in the EA

$$b_\lambda \bar{C}^{\mu\nu\rho\sigma} \log \bar{\Box} \bar{C}_{\mu\nu\rho\sigma} + b_\xi \bar{R} \log \bar{\Box} \bar{R}$$

in the effective action, and the beta functions are

$$\beta_\lambda = -4 b_\lambda \lambda^2 \; , \;\; \beta_\xi = -2 b_\xi \xi^2 \; .$$

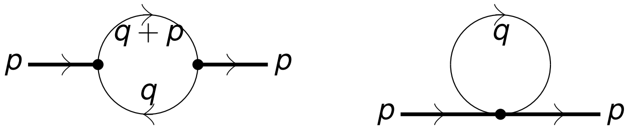

Figure: Diagrams contributing to the two-point function of $\bar{h}$: bubbles (left) and tadpoles (right). The thin line can be the h propagator or one of the ghosts, the thick line is the $\bar{h}$ propagator, with momentum p. The vertices can come from expanding any one among $\mathcal{D}$, $\mathcal{C}$, $\mathcal{V}$, $\mathcal{N}$, $\mathcal{U}$.

The term tr$\mathbb{U}$ in the heat kernel must come from a tadpole. Also some of the tr$R\mathbb{R}$ terms

If one removes those terms, the rest is a bilinear form in $\bar{h}$ that is not the linearization of a covariant expression in $\bar{g}$.

However, there are also infrared contributions to the $log(-p^2)$

No IR divergences in the real world because of m_P.

At high energy one assumes that m_P can be neglected then there would be IR divergences.

IR contributions to $\log p^2$ are not the linearization of a covariant expression in $\bar{g}$ but summing them to the rest we get again a covariant expression.

The terms with $\log \mu^2$ only come from the UV and reproduce the old beta functions.

Beta functions

	Old	New
β_λ	$-\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$	$-\frac{1}{(4\pi)^2}\frac{(1617\lambda-20\xi)\lambda}{90}$
β_ξ	$-\frac{1}{(4\pi)^2}\frac{5(\xi^2-36\lambda\xi+72\lambda^2)}{36}$	$-\frac{1}{(4\pi)^2}\frac{\xi^2-36\lambda\xi-2520\lambda^2}{36}$

Separatrices

Old flow

$$s_1: \quad \xi = \frac{1291 + \sqrt{1637881}}{20}\lambda \approx 128.5\lambda \quad \Rightarrow \omega = -0.0233$$

$$s_2: \quad \xi = \frac{1291 - \sqrt{1637881}}{20}\lambda \approx 0.5601\lambda \quad \Rightarrow \omega = -5.3558$$

New flow

$$s_1: \quad \xi = \frac{569 + \sqrt{386761}}{15}\lambda \approx 79.4\lambda \quad \Rightarrow \omega = -0.03778$$

$$s_2: \quad \xi = \frac{569 - \sqrt{386761}}{15}\lambda \approx -3.53\lambda \quad \Rightarrow \omega = 0.8506$$

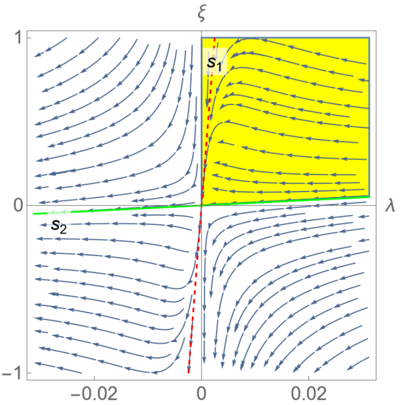

Figure: Left: old flow. Right: new flow.

Main new feature

Asymptotic freedom is possible without the tachyon.

General summary

Whereas in theories with 2-derivative kinetic terms the μ-running and physical running agree in the high energy limit, in theories with 4 derivatives it is not necessarily so.

If there are IR divergences, the two definitions of running may differ. This did not happen in the scalar model but it happens in gravity.

AF this does not yet mean that the theory is well behaved, because the amplitude still grows like E^4.

Meaning of AF in these theories has to be better understood.

Appendix: the $O(3)$ NLSM

$$L = -\frac{g^2}{2} \frac{(\partial_\mu \varphi)^2}{1 + \frac{\varphi_1^2}{4} + \frac{\varphi_2^2}{4}}$$

the $2 \to 2$ amplitude is

$$\mathcal{M} = g_0^2 s - \frac{g_0^4}{4} \left[I(t)(s+t+u) + I(u)(s-t+u) \right]$$

where

$$I(p^2) = T - p^2 B(p^2)$$

is the unique IR finite combination of

$$T = -i \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + i\epsilon}$$

$$B(p^2) = -i \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 + i\epsilon)((p-q)^2 + i\epsilon)}$$

$$g_R^2(E^2) = g_0^2 + \frac{g_0^4}{2} I(E^2)$$

$$I(p^2) - I(E^2) = \log\left(E^2/p^2\right)$$

Then

$$\mathcal{M} = g^2(E^2)s + \frac{g_R^4}{8\pi}\left(\log\frac{-t^2}{E^2} + \log\frac{-u}{E^2}\right) - \frac{g_R^4}{8\pi}(t-u)\log\frac{t}{u}$$

giving

$$\beta_g = E\frac{\partial g_R}{\partial E} = \frac{g^3}{4\pi}$$

With UV and IR cutoff

$$T = \frac{1}{2\pi}\log(\Lambda^2/k^2)$$

$$p^2 B(p^2) = \frac{1}{2\pi}\log(-p^2/k^2)$$

With dimreg at both ends

$$T = 0$$

$$p^2 B(p^2) = \frac{1}{2\pi}\left[\frac{1}{\epsilon} - \log(-p^2/\mu^2)\right]$$

with dimreg in UV and cutoff in IR

$$T = \frac{1}{2\pi}\left[\frac{1}{\epsilon} - \log(k^2/\mu^2)\right]$$

$$p^2 B(p^2) = \frac{1}{2\pi}\log(-p^2/k^2)$$